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ONE PROPERTY OF ψ-FUNCTION

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In [1] my father Krassimir Atanassov has defined the ψ-function (see also [2]). Its definition is the following. For

$$n = \sum_{i=1}^{m} a_i . 10^{m-i} \equiv a_1 a_2 \ldots a_m ,$$

where a_i is a natural number and $0 \le a_i \le 9$ $(1 \le i \le m)$ let

$$\varphi(n) = \begin{cases} 0 & , \text{ if } n = 0 \\ \sum_{i=1}^{m} a_i , & \text{otherwise} \end{cases}$$

and for the sequence of functions $\varphi_0, \varphi_1, \varphi_2, \ldots$,

$$\varphi_0(n) = n, \quad \varphi_{l+1}(n) = \varphi(\varphi_l(n)),$$

let the function ψ be defined by $\psi(n) = \varphi_l(n)$, in which

$$\varphi_{l+1}(n) = \varphi_l(n).$$

In this paper, we shall determine all natural numbers k and n, for which

$$\psi(n)^k = n. \qquad (1)$$

Obviously, the natural number n must be the k-th power of some natural number m, for which $0 \le m \le 9$.

The solutions of (1) for $2 \le k \le 7$ are the following:

$\psi(81)^2 = 81$; $\psi(512)^3 = 512$, $\psi(729)^3 = 729$; $\psi(256)^4 = 256$, $\psi(2401)^4 = 2401$, $\psi(6561)^4 = 6561$; $\psi(32768)^5 = 32768$, $\psi(59049)^5 = 59049$; $\psi(531441)^6 = 531441$; $\psi(128)^7 = 128$, $\psi(16384)^7 = 16384$, $\psi(78125)^7 = 78125$; $\psi(823543)^7 = 823543$, $\psi(2097152)^7 = 2097152$, $\psi(4782969)^7 = 4782969$.

The following assertion is valid.

THEOREM: All solutions $\langle k, n \rangle$ of (1) are:

(a) $\langle 1, 1 \rangle$, $\langle 1, 3 \rangle$, $\langle 1, 6 \rangle$;

(b) $\langle l, 0 \rangle$ for every natural number l;

(c) $\langle 3.l+1, 4^{3.l+1} \rangle$ and $\langle 3.l+1, 7^{3.l+1} \rangle$ for every natural number l, if $\psi(n) = 4$ and $\psi(n) = 7$, respectively;

(d) $\langle 6.l+1, 5^{6.l+1} \rangle$ for every natural number l, if $\psi(n) = 5$;

(e) $\langle 2.l+1,\ 8^{2.l+1}\rangle$ for every natural number l, if $\psi(n) = 8$;

(f) $\langle l,\ 9^{l}\rangle$ for every natural number l, if $\psi(n) = 9$.

PROOF: The validity of (a) and (b) is obvious.

(d) Let $\psi(n) = 5$.

When $l = 0$, obviously $\psi(n)^{1} = 5$. Let us assume that $\langle 6.l+1,\ 5^{6.l+1}\rangle$ is a solution of (1) for some l, i.e.

$$\psi(5^{6.l+1})^{6.l+1} = 5^{6.l+1}.$$

Therefore

$$\psi(5^{6.l+1}) = 5. \qquad (2)$$

Then we construct the number $5^{6.l+7}$ and then

$\psi(5^{6.l+7}) = \psi(5^{6.l+1}.5^{6})$

(from the formula

$$\psi(a.b) = \psi(\psi(a).\psi(b)), \qquad (3)$$

see [1])

$= \psi(\psi(5^{6.l+1}).\psi(5^{6}))$

(from (2))

$= \psi(5.\psi(5^{6})) = \psi(\psi(5).\psi(5^{6}))$

(from (3))

$= \psi(5^{7}) = 5.$

Therefore $\psi(5^{6.l+7})^{6.l+7} = 5^{6.l+7}$,

i.e. $\langle 6.l+1,\ 5^{6.l+1}\rangle$ is a solution.

The other cases are proved analogically.

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[1] Atanassov K., An arithmetic function and some of its applications., Bull. of Number Theory and Related Topics, Vol. IX (1985), No. 1, 18-27.

[2] Shannon A., Horadam A., Generalized staggered sums, The Fibonacci Quarterly, Vol. 29 (1991), No. 1, 47-51.

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