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A COMBINATORIAL SUMMATION IDENTITY FOR POLYGONAL NUMBERS

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The familiar triangular and square numbers are the numbers of the sequences $\{n(n+1)/2\}_{n=1}^{\infty}$ and $\{n^2\}_{n=1}^{\infty}$ respectively. They are special cases of types of numbers called polygonal numbers. Here we speak of $(k+2)$-gonal numbers where $k = 1, 2, 3, \ldots$ Thus when $k = 1, 2$ we have the triangular and square numbers respectively. We denote the nth $(k+2)$-gonal number by $P_{n,k}$. The number $P_{n,k}$ is obtained by summing the first n terms of an arithmetic progression with first term 1 and common difference k. Thus

$$P_{n,k} = \frac{n}{2}(k(n-1)+2). \tag{1}$$

Further information about polygonal numbers may be found for example in [1] and [2].

Our aim here is to prove a summation identity, believed to be new, involving binomial coefficients and powers of polygonal numbers. To assist in the proof we need the following lemma which occurs as identities (1.13) and (1.14) in [3].

Lemma For nonnegative integers j, n

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}i^j = \begin{cases} 0, 0 \le j < n, \\ (-1)^n n!, j = n, \\ \dfrac{(-1)^n n(n+1)!}{2}, j = n+1. \end{cases}$$

Theorem For integers k, m, n with $k \geq 1$, $m \geq 1$, $n \geq m$

$$(2) \qquad \sum_{i=0}^{2m-1}(-1)^i\binom{2m-1}{i}P^m_{n+m-i,k} = \frac{(2m)!k^{m-1}(nk+1)}{2^m}.$$

Proof. Using (1) the left side of (2) is

$$(3) \qquad \frac{1}{2^m}\sum_{i=0}^{2m-1}(-1)^i\binom{2m-1}{i}(n+m-i)^m(k(n+m-i-1)+2)^m$$

and this can be regarded as a polynomial in k of degree m. Now for $m = 1$ (2) follows directly from (1) so consider only $m \geq 2$.

For $0 \leq r \leq m - 2$ differentiating (3) r times with respect to k and putting $k = 0$ yields

$$(4) \qquad \frac{m!}{2^r(m-r)!}\sum_{i=0}^{2m-1}(-1)^i\binom{2m-1}{i}(n+m-i)^m(n+m-i-1)^r.$$

The product of the last two factors in (4) can be regarded as a polynomial in i of degree $m + r \leq 2m - 2$. Thus the Lemma shows that (4) is zero for each $0 \leq r \leq m - 2$. Hence k^{m-1} is a factor of (3) and so (3) has the form

$$(5) \qquad k^{m-1}(Ak+B),$$

where A and B are functionsof m and n which we now determine. The coefficient of k^m in (3) is

$$\frac{1}{2^m}\sum_{i=0}^{2m-1}(-1)^i\binom{2m-1}{i}(n+m-i)^m(n+m-i-1)^m$$

$$=\frac{1}{2^m}\sum_{i=0}^{2m-1}(-1)^i\binom{2m-1}{i}(i^{2m}-m(2m+2n-1)i^{2m-1}+\ldots).$$

Using the Lemma this sum becomes

$$\frac{1}{2^m}\left(\frac{(-1)^{2m-1}(2m-1)(2m)!}{2}-m(2m+2n-1)(-1)^{2m-1}(2m-1)!\right)$$

and this simplifies to give

(6)
$$A = \frac{(2m)!n}{2^m}.$$

Similarly we find

(7)
$$B = \frac{(2m)!}{2^m}$$

and this completes the proof of the Theorem. ❐

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