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ON ONE GENERALIZATION OF THE FIBONACCI SEQUENCE
Part VI: SOME OTHER EXAMPLES

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The paper is the next part of our series related to application of matrix methods in the research on the Fibonacci sequences (see [1-5]). Here we shall use without definitions the notations introduced there and we shall give, new examples.

Let the Fibonacci sequence be given in the form:

$$X_{n+1} = A.X_n \quad (n = 0, 1, 2, \ldots), \tag{1}$$

where X is (M x 1)-matrix and A is a (M x M)-matrix.

THEOREM 1: If all eigenvalues $\mu_1, \mu_2, \ldots, \mu_M$ of A are different, then there exists a sequence with complex numbers u_0, $u_1, \ldots,$ each member of which is dependent on A (but independent of X_0) and complex (M x 1)-matrices Q_0, $Q_1, \ldots, Q_{M-1}$ dependent on A and X_0 with the following property:

$$X_{n+M} = \sum_{K=0}^{M-1} u_{n+K} . Q_K \quad (n = 0, 1, \ldots). \tag{2}$$

If A is regular one (2) is valid for negative integers n, too.

Proof: Let us put:

$$u_n = \sum_{r=1}^{M} C_r . \mu_r^n \quad (n = 0, 1, \ldots)$$

where $C_1, C_2, \ldots, C_M$ are arbitrary complex numbers and all they are $\neq 0$.

Directly it can be checked that (2) is valid for $0 \leq n \leq M - 1$.

Let the characteristic equation $\det(A - \mu.E) = 0$ have the form:

$$\mu^M = \sum_{s=0}^{M-1} p_s . \mu^s ,$$

where $p_0, p_1, \ldots, p_{M-1}$ are real (complex) numbers, if A is a real (complex) matrix. Then for every natural number n:

$$\mu_j^{n+M} = \sum_{s=0}^{M-1} p_s . \mu_j^{n+s} ,$$

from where

$$X_{n+M} = \sum_{s=0}^{M-1} p_s . X_{n+s}$$

(cf. Theorem 2 [4])).

Let Q_0, Q_1, ..., Q_{M-1} be determined such that (2) is valid for X_{n+M} (n = 0, 1, ..., M - 1). Using the method of the induction, we shall show that (2) is valid for all other $n \geq M$.

From (2) we obtain:

$$\sum_{k=0}^{M-1} u_{n+1+k} . Q_k = \sum_{k=0}^{M-1} Q_k . (\sum_{s=0}^{M-1} p_s . u_{n-M+1+k+s})$$

$$= \sum_{s=0}^{M-1} p_s . (\sum_{k=0}^{M-1} Q_k . u_{n-M+1+k+s})$$

$$= \sum_{s=0}^{M-1} p_s . X_{n+1+s} = X_{n+M+1},$$

with which the assertion is proved for n+1.

EXAMPLE 1: The Fibonacci sequence $\alpha_{n+2} = \alpha_{n+1} + \alpha_n$ can be represented in the form:

$$\begin{bmatrix} \alpha_{n+1} \\ \alpha_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} . \begin{bmatrix} \alpha_n \\ \alpha_{n+1} \end{bmatrix}.$$

Here $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $\det(A - \mu.E) = 0$ has the form:

$$\mu^2 - \mu - 1 = 0,$$

from where $\mu_1 = (1 + \sqrt{5})/2$, $\mu_2 = (1 - \sqrt{5})/2$, $\mu_1 . \mu_2 = -1$.

In the particular case ($F_n = \alpha_n$, $F_0 = 0$, $F_1 = 1$):

$$F_n = (\mu_1^n - \mu_2^n)/\sqrt{5} \text{ and } F_{n-1} = (\mu_1^{-1} . \mu_1^n - \mu_2^{-1} . \mu_2^n)/\sqrt{5},$$

from where

$$\mu_1^n = ((1 + \sqrt{5})/2)^n = -((1 - \sqrt{5})/2)^{-1} . F_n + F_{n-1}$$

$$\mu_2^n = ((1 - \sqrt{5})/2)^n = -((1 + \sqrt{5})/2)^{-1} . F_n + F_{n-1}$$

Let the initial conditions are $X_0 = \begin{bmatrix} a \\ b \end{bmatrix}$. Then

$$X_n = \begin{bmatrix} b \\ a+b \end{bmatrix} . F_n + \begin{bmatrix} a \\ b \end{bmatrix} . F_{n-1}.$$

Let the (1) be given as above. Let B'_1 and B'_2 be fixed complex (M x 1)-matrices and let $\Phi_k(n)$ be complex (M x 1)-matrices for 1

$\le k \le s$, which represent periodical functions of n. Let μ_1, μ_2, $\dots$, μ_M be the eigenvalues of A, and let F_n be the ordinary Fibonacci numbers.

THEOREM 2: For every natural number n:

$$X_n = F_{n+m_1}.B'_1 + F_{n+m_2}.B'_2 + \sum_{k=1}^{r} \Phi_k(n), \qquad (3)$$

where m_1 and m_2 are some constants, if and only if $\mu_{1,2} = (1 \pm \sqrt{5})/2$ and all other eigenvalues of A μ_k have the form: $\mu_k = \exp(i.\alpha_k)$, where α_k are different real numbers and $i = \sqrt{-1}$; and eventually, $\mu_M = 0$.

Moreover, (3) can be represented in the form:

$$X_n = F_n.B_1 + F_{n-1}.B_2 + \sum_{k=1}^{s} (C_k.\cos(\alpha_k.n) + D_k.\sin(\alpha_k.n)), \qquad (4)$$

where B_1, B_2, C_k, D_k $(1 \le k \le s)$ are fixed $(M \times 1)$-matrices.

We shall denote that in the assertion (3) and (4) must be valid for arbitrary initial values of X_0. If X_0 has a fixed initial value, then the necessary and sufficient condition (3) and (4) to be valid for any initial value of X_0 is that $\mu_{1,2} = (1 \pm \sqrt{5})/2$ to be among the eigenvalues of A.

The proof of the last theorem it follows from Theorem 1 [4] and Theorem 2 [1].

EXAMPLE 2 (see [7]): The eigenvalues of A for the scheme

$$\begin{cases} \alpha_0 = a,\ \alpha_1 = b,\ \beta_0 = c,\ \beta_1 = d, \\ \alpha_{n+2} = \alpha_{n+1} + \beta_n \\ \beta_{n+2} = \beta_{n+1} + \alpha_n \end{cases}$$

are $\mu_{1,2} = (1 \pm \sqrt{5})/2$, $\mu_{3,4} = \exp(\pm i.\pi/3)$.

EXAMPLE 3 (see [6]): The eigenvalues of A for the scheme

$$\begin{cases} \alpha_0 = a,\ \alpha_1 = b,\ \beta_0 = c,\ \beta_1 = d, \\ \alpha_{n+2} = \beta_{n+1} + \beta_n \\ \beta_{n+2} = \alpha_{n+1} + \alpha_n \end{cases}$$

are $\mu_{1,2} = (1 \pm \sqrt{5})/2$, $\mu_{3,4} = \exp(\pm 2.i.\pi/3)$.

EXAMPLE 4: The eigenvalues of A for the scheme

$$\begin{cases} \alpha_0 = a, \ \alpha_1 = b, \ \beta_0 = c, \ \beta_1 = d, \\ \alpha_{n+2} = \alpha_{n+1} + \beta_n \\ \beta_{n+2} = \alpha_{n+1} + \alpha_n \end{cases}$$

are $\mu_{1,2} = (1 \pm \sqrt{5})/2$, $\mu_{3,4} = \pm i$.

EXAMPLE 5: The eigenvalues of A for the scheme

$$\begin{cases} \alpha_0 = a, \ \alpha_1 = b, \ \beta_0 = c, \ \beta_1 = d, \\ \alpha_{n+2} = \beta_{n+1} + \alpha_n \\ \beta_{n+2} = \alpha_{n+1} + \alpha_n \end{cases}$$

are $\mu_{1,2} = (1 \pm \sqrt{5})/2$, $\mu_3 = -1$, $\mu_4 = 0$.

EXAMPLE 6: The eigenvalues of A for the scheme

$$\begin{cases} \alpha_0 = a, \ \alpha_1 = b, \ \alpha_2 = c, \\ \alpha_{n+3} = 2.\alpha_{n+1} + \alpha_n \end{cases}$$

are $\mu_{1,2} = (1 \pm \sqrt{5})/2$, $\mu_3 = -1$.

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