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ON ONE GENERALIZATION OF THE FIBONACCI SEQUENCE
Part V: SOME EXAMPLES

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The paper is the fifth part of our series related to application of matrix methods in the research on the Fibonacci sequences (see [1-4]). Here we shall use without definitions the notations introduced there and we shall give, as examples, the formulas of the generalizations of the 2-Fibonacci sequences introduced in [5-8].

Let a, b, c, d, p, q, r and s be given real numbers.

Here we shall generalize the schemes from [5] to the following:

Scheme 1:

$$\begin{cases} \alpha_0 = a, \ \alpha_1 = b, \ \beta_0 = c, \ \beta_1 = d, \\ \alpha_{n+2} = p.\alpha_{n+1} + q.\beta_n & n \in N \\ \beta_{n+2} = r.\beta_{n+1} + s.\alpha_n \end{cases} \tag{1}$$

Scheme 2:

$$\begin{cases} \alpha_0 = a, \ \alpha_1 = b, \ \beta_0 = c, \ \beta_1 = d, \\ \alpha_{n+2} = p.\beta_{n+1} + q.\beta_n & n \in N \\ \beta_{n+2} = r.\alpha_{n+1} + s.\alpha_n \end{cases} \tag{2}$$

Scheme 3:

$$\begin{cases} \alpha_0 = a, \ \alpha_1 = b, \ \beta_0 = c, \ \beta_1 = d, \\ \alpha_{n+2} = p.\beta_{n+1} + q.\alpha_n & n \in N \\ \beta_{n+2} = r.\alpha_{n+1} + s.\beta_n \end{cases} \tag{3}$$

The fourth scheme is a trivial one.

Sequentially, we shall apply the methods from [1-4] over the three schemes.

Let Scheme 1 be given. We rewrite it in the form:

$$\begin{bmatrix} \alpha_{n+1} \\ \alpha_{n+2} \\ \beta_{n+1} \\ \beta_{n+2} \end{bmatrix} = A. \begin{bmatrix} \alpha_n \\ \alpha_{n+1} \\ \beta_n \\ \beta_{n+1} \end{bmatrix},$$

i.e. $X_{n+1} = A.X_n$, where for every natural number $k \geq 0$:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & p & q & 0 \\ 0 & 0 & 0 & 1 \\ s & 0 & 0 & r \end{bmatrix} \quad \text{and} \quad X_k = \begin{bmatrix} \alpha_k \\ \alpha_{k+1} \\ \beta_k \\ \beta_{k+1} \end{bmatrix}$$

The characteristic equation is

$$P(\mu) = 0, \tag{4}$$

where

$$P(\mu) \equiv \det(A - \mu.E) = \mu^4 - (p + r).\mu^3 + p.r.\mu^2 - q.s$$

where E is the single matrix. For the solutions of (4) it follows from Theorem 2 [1] and Theorem 1 [4] that there exist the following cases:

a) four different roots:

$$X_n = B.\mu_1^n + C.\mu_2^n + D.\mu_3^n + F.\mu_4^n, \tag{5}$$

where B, C, D and F are (4x1)-matrices with complex elements.

b) two equal roots ($\mu_3 = \mu_4$):

$$X_n = B.\mu_1^n + C.\mu_2^n + (D.n + F).\mu_3^n, \tag{6}$$

c) three equal roots ($\mu_2 = \mu_3 = \mu_4$):

$$X_n = B.\mu_1^n + (C.n^2 + D.n + F).\mu_2^n, \tag{7}$$

d) four equal roots ($\mu_1 = \mu_2 = \mu_3 = \mu_4$):

$$X_n = (B.n^3 + C.n^2 + D.n + F).\mu_1^n, \tag{8}$$

e) two tuples of equal roots ($\mu_1 = \mu_2 \neq \mu_3 = \mu_4$):

$$X_n = (B.n + C).\mu_1^n + (D.n + F).\mu_3^n. \tag{9}$$

From (4) and Theorem 2 [4] it follows the validity of

THEOREM 1: For every X_0 and for every natural number $n \geq 0$:

$$X_{n+4} = (p + r).X_{n+3} - p.r.X_{n+2} + q.s.X_n.$$

In particular:

$$\begin{cases} \alpha_{n+4} = (p + r).\alpha_{n+3} - p.r.\alpha_{n+2} + q.s.\alpha_n. \\ \beta_{n+4} = (p + r).\beta_{n+3} - p.r.\beta_{n+2} + q.s.\beta_n. \end{cases}$$

All other schemes from [8] are solved similarly.

Let J = L = 1. From Theorem 1 [2] it follows that $H_n.X_0 = 0$, where $H_n = P.A^n + Q + \sum_{k=0}^{n} S.A^k$ is transformed to

$$G \equiv Q.(E - A) + S = 0,$$

$$H_0 = P + Q + S,$$

i.e., $S = Q.(A - E)$ and $P = -Q.A$.

EXAMPLES:

1. Let $Q = \langle 0, 1, q, 0 \rangle$. Then

$$S = Q.(A - E) = \langle 0, p-1, 0, q \rangle$$

$$P = -Q.A = \langle 0, -p, -q, -q \rangle$$

and from $H_n.X_0 = 0$ it follows that

$$-p.\alpha_{n+1} - q.\beta_n - q.\beta_{n+1} + \alpha_1 + q.\beta_0 + \sum_{k=0}^{n} ((p-1).\alpha_{k+1} + q.\beta_{k+1}) = 0.$$

and from (1) it follows that:

$$-\alpha_{n+2} - q.\beta_{n+1} + \alpha_1 + q.\beta_0 + \sum_{k=0}^{n} ((p-1).\alpha_{k+1} + q.\beta_{k+1}) = 0,$$

i.e., only coeffitients p and q there exist here (r and s do not exist).

2. Let $Q = \langle p, -1, 0, 0 \rangle$. Then

$$S = Q.(A - E) = \langle -p, 1, -q, 0 \rangle$$

$$P = -Q.A = \langle 0, 0, q, 0 \rangle$$

and from $H_n.X_0 = 0$ it follows that

$$q.\beta_{n+1} + p.\alpha_1 - \alpha_2 + \sum_{k=1}^{n+1} (-p.\alpha_k + \alpha_{k+1} - q.\beta_{k+1}) = 0.$$

Therefore, only coeffitients p and q there exist also (r and s do not exist).

3. Let $Q = \langle 0, 0, r, -1 \rangle$. Then

$$S = Q.(A - E) = \langle -s, 0, -r, 1 \rangle$$

$$P = -Q.A = \langle s, 0, 0, 0 \rangle$$

and

$$s.\alpha_n + r.\beta_0 - \beta_1 + \sum_{k=0}^{n} (-s.\alpha_k - r.\beta_k + \beta_{k+1}) = 0.$$

Here coeffitients p and q do not exist.

4. Let $Q = \langle -s, 0, 0, -1 \rangle$. Then

$$S = Q.(A - E) = \langle 0, -s, 0, 1-r \rangle$$

$$P = -Q.A = \langle s, s, 0, r \rangle$$

and

$$s.\alpha_n + s.\alpha_{n+1} + r.\beta_{n+} - s.\alpha_0 - \beta_1 + \sum_{k=0}^{n} (-s.\alpha_{k+1} + (1-r).\beta_{k+1}) = 0.$$

and from (1) it follows that:

$$\beta_{n+2} + s.\alpha_{n+1} - s.\alpha_0 - \beta_1 + \sum_{k=0}^{n} (-s.\alpha_{k+1} + (1-r).\beta_{k+1}) = 0,$$

i.e., coeffitients p and q do not exist also.

Let Scheme 2 be given. We rewrite it in the form:

$$X_{n+1} = A.X_n,$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & q & p \\ 0 & 0 & 0 & 1 \\ s & r & 0 & 0 \end{bmatrix} \quad \text{and} \quad X_n = \begin{bmatrix} \alpha_n \\ \alpha_{n+1} \\ \beta_n \\ \beta_{n+1} \end{bmatrix}$$

The characteristic equation (4) has a left hand

$P(\mu) \equiv \det(A - \mu.E) = \mu^4 - p.r.\mu^2 - (q.r + p.s).\mu - q.s.$

The common member has one of the forms from (5)-(9).

THEOREM 2: For every X_0 and for every natural number $n \geq 0$:

$$X_{n+4} = p.r.X_{n+2} + (q.r + s.p).X_{n+1} + q.s.X_n .$$

The proof is similar to the above one.

For α- and β-members we obtain:

$$\begin{cases} \alpha_{n+4} = p.r.\alpha_{n+2} + (q.r + s.p).\alpha_{n+1} + q.s.\alpha_n \\ \beta_{n+4} = p.r.\beta_{n+2} + (q.r + s.p).\beta_{n+1} + q.s.\beta_n \end{cases}$$

EXAMPLES:

1. Let $Q = \langle 0, -1, p, 0\rangle$. Then

$$S = Q.(A - E) = \langle 0, 1, -(p + q), 0\rangle$$
$$P = -Q.A = \langle 0, 0, q, 0\rangle$$

and from $H_n.X_0 = 0$ it follows that

$$q.\beta_{n+1} - \alpha_2 + p.\beta_1 + \sum_{k=0}^{n} (\alpha_{k+2} - (p + q).\beta_{k+1}) = 0,$$

i.e.

$$q.\beta_{n+1} - \alpha_2 + p.\beta_1 + q.\sum_{k=0}^{n} (\beta_k - \beta_{k+1}) = 0,$$

Therefore, only coeffitients p and q there exist.

2. Let $Q = \langle 0, -1, -q, 0\rangle$. Then

$$S = Q.(A - E) = \langle 0, 1, 0, -(p + q)\rangle$$
$$P = -Q.A = \langle 0, 0, q, p + q\rangle$$

and

$$q.\beta_n + (p + q).\beta_{n+1} - \alpha_1 - q.\beta_0 + \sum_{k=0}^{n} (\alpha_{k+1} - (p + q).\beta_{k+1}) = 0.$$

and from (2) it follows that:

$$\alpha_{n+2} + q.\beta_{n+1} - \alpha_1 - q.\beta_0 + \sum_{k=0}^{n} (\alpha_{k+1} - (p + q).\beta_{k+1}) = 0.$$

3. Let $Q = \langle -s, 0, 0, -1\rangle$. Then

$$S = Q.(A - E) = \langle 0, -(s + r), 0, 1\rangle$$
$$P = -Q.A = \langle -s, s + r, 0, 0\rangle$$

and

$$s.\alpha_n + (r + s).\alpha_{n+1} - s.\alpha_0 - \beta_1 + \sum_{k=0}^{n} (-(r + s).\alpha_{k+1} - \beta_{k+1}) = 0.$$

4. Let $Q = \langle r, 0, 0, -1\rangle$. Then

$$S = Q.(A - E) = \langle -(r + s), 0, 0, 1\rangle$$
$$P = -Q.A = \langle s, 0, 0, 0\rangle$$

and

$$s.\alpha_n + r.\alpha_0 - \beta_1 + \sum_{k=0}^{n} (-(r + s).\alpha_k + \beta_{k+1}) = 0.$$

and from (2) it follows that:

$$\beta_{n+2} + s.\alpha_{n+1} - s.\alpha_0 - \beta_1 + \sum_{k=0}^{n} (-s.\alpha_{k+1} + (1-r).\beta_{k+1}) = 0,$$

i.e., coeffitients p and q do not exist also.

Let Scheme 3 be given. We rewrite it in the form:

$$X_{n+1} = A.X_n,$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ q & 0 & 0 & p \\ 0 & 0 & 0 & 1 \\ 0 & r & s & 0 \end{bmatrix}.$$

The characteristic equation (4) has a left hand

$$P(\mu) \equiv \det(A - \mu.E) = \mu^4 - (p.r + q + s).\mu^2 - q.s.$$

The common member has one of the forms from (5)-(9).

THEOREM 3: For every X_0 and for every natural number $n \geq 0$:

$$X_{n+4} = (p.r + s + q).X_{n+2} - q.s.X_n.$$

The proof is similar to the above one.

For α- and β-members we obtain:

$$\begin{cases} \alpha_{n+4} = (p.r + s + q).\alpha_{n+2} - q.s.\alpha_n. \\ \beta_{n+4} = (p.r + s + q).\beta_{n+2} - q.s.\beta_n. \end{cases}$$

EXAMPLES:

1. Let $Q = \langle -q, -1, 0, 0\rangle$. Then

$$S = Q.(A - E) = \langle 0, -q + 1, 0, -p\rangle$$
$$P = -Q.A = \langle q, q, 0, p\rangle$$

and

$$q.\alpha_n + q.\alpha_{n+1} + p.\beta_{n+1} - q.\alpha_0 - \alpha_1 + \sum_{k=0}^{n} ((1 - q).\alpha_{k+1} - p.\beta_{k+1}) = 0,$$

i.e.

$$\alpha_{n+2} + q.\alpha_{n+1} - q.\alpha_0 - \alpha_1 + \sum_{k=0}^{n} ((1 - q).\alpha_{k+1} - p.\beta_{k+1}) = 0.$$

2. Let $Q = \langle 0, -1, p, 0\rangle$. Then

$$S = Q.(A - E) = \langle -q, 1, -p, 0\rangle$$
$$P = -Q.A = \langle q, 0, 0, 0\rangle$$

and

$$q.\alpha_n - \alpha_1 + p.\beta_0 + \sum_{k=0}^{n} (-q.\alpha_k + \alpha_{k+1} - p.\beta_{k+1}) = 0.$$

3. Let $Q = \langle r, 0, 0, -1\rangle$. Then

$$S = Q.(A - E) = \langle -r, 0, -s, 1\rangle$$

$$P = -Q.A = \langle 0, 0, s, 0\rangle$$

and

$$s.\beta_n + r.\alpha_0 - \beta_1 + \sum_{k=0}^{n} (-r.\alpha_k - s.\beta_k + \beta_{k+1}) = 0.$$

4. Let $Q = \langle 0, 0, -s, -1\rangle$. Then

$$S = Q.(A - E) = \langle 0, -r, 0, 1-s\rangle$$

$$P = -Q.A = \langle 0, r, s, s\rangle$$

and

$$r.\alpha_{n+1} + s.\beta_n + s.\beta_{n+1} - s.\beta_0 - \beta_1 + \sum_{k=0}^{n} (-r.\alpha_{k+1} + (1-s).\beta_{k+1}) = 0,$$

and from (3) it follows that:

$$\beta_{n+2} + s.\beta_{n+1} - s.\beta_0 - \beta_1 + \sum_{k=0}^{n} (-r.\alpha_{k+1} + (1 - s).\beta_{k+1}) = 0.$$

REFERENCES:

[1] Georgiev P., Atanassov K., On one generalization of the Fibonacci sequence. Part I: Matrix representation. Bulletin of Number Theory, Vol. XVI, 1992, 67-73.

[2] Georgiev K., Atanassov K., On one generalization of the Fibonacci sequence. Part II: Some relations with arbitrary initial values. Bulletin of Number Theory, Vol. XVI, 1992, 75-82.

[3] Georgiev K., Atanassov K., On one generalization of the Fibonacci sequence. Part III: Some relations with fixed initial values. Bulletin of Number Theory, Vol. XVI, 1992, 83-92.

[4] Georgiev P., Atanassov K., On one generalization of the Fibonacci sequence. Part IV: Multiplicity roots of the characterestic equation. see pp. 3-7.

[5] Atanassov K., Atanassova L., Sasselov D. A new perspective to the generalization of the Fibonacci sequence, The Fibonacci Quarterly Vol. 23 (1985), No. 1, 21-28.

[6] Atanassov K. On a second new generalization of the Fibonacci sequence. The Fibonacci Quarterly Vol. 24 (1986), No. 4, 362-365.

[7] Lee J.-Z., Lee J.-S., Some properties of the generalization of the Fibonacci sequence. The Fibonacci Quarterly Vol. 25 (1987), No. 2, 111-117.

[8] Atanassov K., Hlebarska J., Mihov S. Recurrent formulas of the generalized Fibonacci and Tribonacci sequences. The Fibonacci Quarterly Vol. 30, (1992), No. 1, 77-79.

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