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ON ONE GENERALIZATION OF THE FIBONACCI SEQUENCE

Part IV: MULTIPLICITY ROOTS OF THE CHARACTERISTIC EQUATION

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The paper is the fourth part of our series related to application of matrix methods in the research on the Fibonacci sequences (see [1-3]). Here we shall use without definitions the notations introduced there.

THEOREM 1: Let A be a (M x M)-matrix with eigenvalues μ_1, μ_2, ..., μ_k and let μ_i has a multiplicity m_i ($1 \le i \le k$) such that $\sum_{i=1}^{k} m_i = M$. Then the generalized Fibonacci sequence (for every natural number $n \ge 0$)

$$X_{n+1} = A.X_n$$

has a n-th term with the form:

$$X_n = \sum_{i=1}^{k} P_{m_i - 1}(n).\mu_i^n, \tag{1}$$

where $P_{m_i}(n)$ is a matrix-polynomial with a degree $\le m_i$ ($1 \le i \le k$).

Proof: From Jordan's theorem (see e.g. [4]) it follows that there exists a square matrix T with $\det(T) \ne 0$, such that

$$A = T.D.T^{-1},$$

where the matrix D has the following block-form

$$D = \begin{bmatrix} M_1 & O & \dots & O \\ O & M_2 & \dots & O \\ & . & . & . \\ O & O & \dots & M_k \end{bmatrix},$$

i.e., D is a (k x k)-matrix, where for $1 \le i \le k$:

$$M_i = \begin{bmatrix} \mu_i & 1 & 0 & \dots & 0 \\ 0 & \mu_i & 1 & \dots & 0 \\ & \cdot & \cdot & \cdot & \\ 0 & 0 & 0 & \dots & \mu_i \end{bmatrix},$$

i.e. M_i is a $(m_i \times m_i)$-matrix.

Then

$$A^n = T.D^n.T^{-1},$$

where

$$D^n = \begin{bmatrix} M_1^n & 0 & \dots & 0 \\ 0 & M_2^n & \dots & 0 \\ \cdot & \cdot & \cdot & \\ 0 & 0 & \dots & M_k^n \end{bmatrix},$$

and

$$M_i^n = \begin{bmatrix} \binom{n}{0}\mu_i^n & \binom{n}{1}\mu_i^{n-1} & \binom{n}{2}\mu_i^{n-2} & \dots & \binom{n}{m_i-1}\mu_i^{n-m_i+1} \\ 0 & \binom{n}{0}\mu_i^n & \binom{n}{1}\mu_i^{n-1} & \dots & \binom{n}{m_i-2}\mu_i^{n-m_i+2} \\ 0 & 0 & \binom{n}{0}\mu_i^n & \dots & \binom{n}{m_i-3}\mu_i^{n-m_i+3} \\ & \cdot & \cdot & \cdot & \\ 0 & 0 & 0 & & \binom{n}{0}\mu^n \end{bmatrix}$$

Therefore, M_i^n has the form

$$M_i^n = S_{m_i-1}(n).\mu_i^n,$$

where $S_{m_i-1}(n)$ is a $(m_i \times m_i)$-matrix polynomial with a degree $\leq m_i - 1$, with which the Theorem is proved.

We must note that if $\det(A) \neq 0$, then formula (1) is valid for integer numbers $n < 0$, too.

We shall introduce an example which illustrates the above assertion.

Let $\alpha_0 = b$, $\beta_0 = 2.a$ and

$$\begin{cases} \alpha_{n+1} = 2.\alpha_n + \beta_n \\ \beta_{n+1} = 2.\beta_n \end{cases}$$

Then we construct the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

with $\mu_1 = 2$ and multiplicity $m_1 = 2$. From Theorem 1 we obtain that

$$\begin{cases} \alpha_n = (a.n + b).2^n \\ \beta_n = 2^{n+1}.a \end{cases}$$

From the Hamilton-Cayley's theorem (see, e.g. [4]) it follows the validity of

THEOREM 2: The (n+m)-th term of every sequence

$$X_{n+1} = A.X_n$$

can be represented in the form

$$X_{n+m} = \sum_{i=0}^{m-1} p_i.X_{n+i},$$

where $p_0, p_1, \ldots, p_{m-1}$ are some complex numbers, that depend only on A, but do not depend $X_0, X_1, \ldots$.

More interesting is the case of a sequence (a generalized Fibonacci sequence) with the form (where n is an arbitrary integer ≥ 0):

$$X_{n+1} = A.X_n + B_m(n),$$

where A and X are (M x M)- and (M x 1)-matrics, respectively and $B_m(n)$ is a (M x 1)-matrix polynomial with a degree $\leq$ m.

Let $E = A^0$ be the (M x M)-single matrix.

THEOREM 3: Let for the above matrix A be valid that $\det(A - E) \neq 0$. Then X_n has the form

$$X_n = A^n.C + F_m(n),$$

where C is a (M x 1)-matrix, $F_m(n)$ is a (M x 1)-matrix polynomial with a degree $\leq$ m and

$$A^n.C = \sum_{i=1}^{k} P_{m_i - 1}(n).\mu_i^n,$$

as in Theorem 1.

This assertion can be proved by induction about m. We must determine a matrix-polynomial $F_m(n)$ which satisfies

$$F_m(n+1) - A.F_m(n) = B_m(n), \qquad (2)$$

and for it we must show that it can be represented by a recurrent formula. Really, this is a fact because for the coefficients $F_m^{(0)}$ and $B_m^{(0)}$ before the highest degree n^m we have:

$$F_m^{(0)}(n+1)^m - A.F_m^{(0)}.n^m = B_m^{(0)}.n^m + C_{m-1}(n),$$

where $C_{m-1}(n)$ is a matrix-polynomial with a degree $\leq$ m -1. Then

$$(E - A).F_m^{(0)} = B_m^{(0)}$$

and therefore

$$F_m^{(0)} = -(A - E)^{-1}.B_m^{(0)}.$$

Hence we have means for calculation of $F_m^{(0)}$ i.e., these coeffitients can be represented by a recurrent formula.

We must note that if $\det(A) \neq 0$, then the above assertion is valid for integer numbers $n < 0$, too.

Finally, we shall give the following example.

Let

$$\alpha_{n+2} = \alpha_{n+1} + \alpha_n + a.n + b$$

Then from

$$\begin{bmatrix} \alpha_{n+1} \\ \beta_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} . \begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix} + \begin{bmatrix} 0 \\ a.n + b \end{bmatrix},$$

and

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

it follows that

$$\begin{cases} \alpha_n = \left(\frac{1 + \sqrt{5}}{2}\right)^n .B + \left(\frac{1 - \sqrt{5}}{2}\right)^n .C - a.n - (a + b), \\ \beta_n = \alpha_{n+1}, \end{cases}$$

where B and C are constants.

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