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IRRATIONAL FACTOR: DEFINITION, PROPERTIES AND PROBLEMS

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Every natural number n has a canonical representation in the form $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, where $p_1, p_2, ..., p_k$ are different prime numbers and $\alpha_1, \alpha_2, ..., \alpha_k \geq 1$ are natural numbers.

Let us take for a given n the new number

$$n' = \prod_{i=1}^{k} p_i^{1/\alpha_i}.$$

It can be easily seen that if for every $i (1 \leq i \leq k)$ $\alpha_i = 1$, then $n' = n$ and n' is a natural number.

On the other hand, if there is at least one $\alpha_i > 1$, then $n' \neq n$ and now n' is not a natural and even not a rational number.

Therefore, in this case n' is an irrational number. Below, n' will be denoted as $IF(n)$ and it will be called "Irrational Factor" of n.

On the other hand, $IF(n)$ is a product of the solutions of the algebraic equations system

$$x_1^{\alpha_1} = p_1$$

$$x_2^{\alpha_2} = p_2$$

$$\dots$$

$$x_k^{\alpha_k} = p_k$$

Therefore, $IF(n)$ is a product of algebraic numbers and hence it is an algebraic number.

<u>THEOREM 1</u>: IF is a multiplicative function.

Indeed, let m and n be two natural numbers for which $(m, n) = 1$. Therefore, if n has the above form and $m = \prod_{i=1}^{l} q_i^{\beta_i}$, where $q_1, q_2, ..., q_k$ are different prime numbers, for every $i (1 \leq i \leq k)$ and for every $j (1 \leq j \leq l) : p_i \neq q_j, and \beta_1, \beta_2, ..., \beta_k \geq 1$ are natural numbers, then

$$IF(n.m) = \prod_{i=1}^{k} p_i^{1/\alpha_i}. \prod_{j=1}^{l} q_j^{1/\beta_j} = IF(n).IF(m).$$

IF is not a monotone function. Its first 40 values are the following

n	IF(n)	n	IF(n)	n	IF(n)	n	IF(n)
1	1	11	11	21	21	31	31
2	2	12	$3.\sqrt{2}$	22	22	32	$\sqrt[5]{2}$
3	3	13	13	23	23	33	33
4	$\sqrt{2}$	14	14	24	$3.\sqrt[3]{2}$	34	34
5	5	15	15	25	$\sqrt{5}$	35	35
6	6	16	$\sqrt[4]{2}$	26	26	36	$\sqrt{6}$
7	7	17	17	27	$\sqrt[3]{3}$	37	37
8	$\sqrt[3]{2}$	18	$2.\sqrt{3}$	28	$7.\sqrt{2}$	38	38
9	$\sqrt{3}$	19	19	29	29	39	39
10	10	20	$5.\sqrt{2}$	30	30	40	$5.\sqrt[3]{2}$

If $k = l, p_1 = q_1, ..., p_k = q_k$ and $\alpha_1 \geq \beta_1, \alpha_2 \geq \beta_2, ...\alpha_k \geq \beta_l$, then $n \geq m$ and $IF(n) \leq IF(m)$. The inequality will be strong if at least one of the inequalities between α_i and $\beta_i (1 \leq i \leq k)$ is strong.

On the other hand, if $k < l, p_1 = q_1, ..., p_k = q_k$ and $\alpha_1 = \beta_1, \alpha_2 = \beta_2, ...\alpha_k = \beta_l$, then $n < m$ and $IF(n) < IF(m)$.

Very interesting, but open, is the question of the relation between n and m when $k < l, p_1 = q_1, ..., p_s = q_s$ for $0 \leq s \leq k$ and there are not restrictions for the relations between $\alpha-$ and $\beta-$ powers.

Obviously, $IF(n) > 1$ for every natural number $n > 1$.

<u>THEOREM 2</u>: For every two natural numbers n and m:

$$IF(n^m) = \sqrt[m]{IF(n)}.$$

Moreover, we must note that for every natural number n there exists a natural number m such that $IF(n)^m$ is a natural number. The minimal such number is $m = GCD(\alpha_1, ..., \alpha_k)$.

For the Moebius function μ (see e.g., [1]) for every natural number n there holds the equality

$$|\mu(n)| = \overline{sg}(n - IF(n)),$$

where for every integer number $n \geq 0$:

$$\overline{sg}(x) = \begin{cases} 0, & \text{if } x > 0 \\ 1, & \text{if } x = 0 \end{cases}$$

Let $J(n) = [\frac{n}{IF(n)}]$.

Therefore

$$J(n) \leq \frac{n}{[IF(n)]} \leq n.$$

From the inequality for every natural number n and every prime number p

$$p.\frac{p^n}{p^{1/n}} < \frac{p^{n+1}}{p^{1/(n+1)}}$$

it follows that

$$p.J(p^n) \leq J)p^{n+1}).$$

The question about the relations between the well-known multiplicative functions φ, ψ, σ (see e.g., [1]) and IF is important, but open in general.

For example, we can calculate that

$$J(3^7) = 1869 > 1458 = \varphi(3^7)$$

and

$$J(7^3) = 179 < 294 = \varphi(7943).$$

However it is not known when the following inequalities are valid for the natural number n:

$$(a) J(n) \leq \varphi(n),$$

$$(b) n - J(n) \geq \varphi(n),$$

$$(c) J(n) + n \leq \psi(n),$$

$$(d) J(n).n \geq \sigma(n),$$

$$(e) \varphi(n).\sigma(n) \leq n^2 - J(n)?$$

REFERENCE:

[1] Chandrasekharan K. Introduction to analytic number theory, Springer-Verlag, Berlin, 1968.