NNTDM 2 (1996) 3, 12-22

MATRICES ASSOCIATED WITH A CLASS OF ARITHMETICAL IDENTITIES

by Pentti Haukkanen

1. Introduction

It is well known that some important arithmetical functions possess the structure of the form

$$\Psi(m,n) = \sum_{d|(m,n)} f(d)g(m/d)h(n/d). \tag{1.1}$$

For example, the classical Ramanujan's sum $C(m,n)$ satisfies

$$C(m,n) = \sum_{d|(m,n)} d\mu(n/d), \tag{1.2}$$

where μ is the Möbius function. The divisor function σ_k satisfies

$$\sigma_k(mn) = \sum_{d|(m,n)} d^k\mu(d)\sigma_k(m/d)\sigma_k(n/d).$$

The divisor function is a member of the class of the so-called specially multiplicative functions. In fact, a multiplicative function f is said to be specially multiplicative [5, 6] if there exists a completely multiplicative function g such that

$$f(mn) = \sum_{d|(m,n)} \mu(d)g(d)f(m/d)f(n/d). \tag{1.3}$$

It is well known [5, 6] that (1.3) holds if, and only if,

$$f(m)f(n) = \sum_{d|(m,n)} g(d)f(mn/d^2). \tag{1.4}$$

In [2], Bourque and Ligh gave a matrix representation of the structure (1.1). We refer to this representation as the representation of type I, see Section 2. We show how this representation easily yields certain types of inverse forms of (1.1), see Section 2.1. These inverse forms also come out naturally from the algebra of Dirichlet convolution of one variable, see Section 2.2.

The fundamental inverse form of (1.3), namely the identity (1.4), does not however follow naturally from the matrix representation of type I. For this reason we construct another matrix representation of the structure (1.1), referred to as the matrix representation of type II, see Section 3. This representation leads to a certain type of inverse form of (1.1), which gives the desired identity (1.4) as a special case, see Section 3.1. This type of inverse form can also be easily verified with aid of the concepts of principal function and Dirichlet convolution of two variables.

We also introduce a unitary analogue of the structure (1.1) and point out that this analogue possesses inverse forms similar to those of the original structure, see Section 4.

We assume that the reader is familiar with the elements of arithmetical functions [1, 5, 6] and matrix algebra.

2. Matrix representation of type I

Let s be an arbitrary but fixed positive integer. Let $\big[\Psi(m,n)\big]$ denote the $s\times s$ matrix whose m,n-entry is $\Psi(m,n)$. Then (1.1) can be written as

$$\big[\Psi(m,n)\big] = \big[g(m/n)\big]\,\Lambda\,\big[h(n/m)\big], \tag{2.1}$$

where $\Lambda = \mathrm{diag}\big(f(1), f(2), \ldots, f(s)\big)$, see [2, Lemma 1]. In particular, (1.2) can be written as

$$\big[C(m,n)\big] = \big[u(m/n)\big]\,\Lambda\,\big[\mu(n/m)\big], \tag{2.2}$$

where $u(m/n)=1$ if $n\,|\,m$, and $=0$ otherwise, and $\Lambda = \mathrm{diag}\big(1,2,\ldots,s\big)$. Further, (1.3) can be written as

$$\big[f(mn)\big] = \big[f(m/n)\big]\,\Lambda\,\big[f(n/m)\big], \tag{2.3}$$

where $\Lambda = \mathrm{diag}\big(\mu(1)g(1), \mu(2)g(2), \ldots, \mu(s)g(s)\big)$.

2.1. Inverse forms via matrix algebra

In derivation of inverse forms we use the following result due to Bourque and Ligh [2, Lemma 2].

LEMMA 1. If f is an arithmetical function with $f(1) \neq 0$, then

$$[f(m/n)]^{-1} = [f^{(-1)}(m/n)],$$

where $[f(m/n)]^{-1}$ denotes the inverse matrix of $[f(m/n)]$ and $f^{(-1)}$ denotes the Dirichlet inverse of the arithmetical function f.

Remark. The Dirichlet inverse $f^{(-1)}$ exists if, and only if, $f(1) \neq 0$.

Example. As $u^{(-1)} = \mu$ and $g^{(-1)} = \mu g$, where g is completely multiplicative, we obtain

$$[u(m/n)]^{-1} = [\mu(m/n)], \ [g(m/n)]^{-1} = [(\mu g)(m/n)].$$

Now, we are in a position to derive inverse forms for the identity (1.1) via matrix algebra. To be more illustrative, we derive inverse forms in the special cases (1.2) and (1.3). In fact, multiplication of (2.2) by $[\mu(n/m)]^{-1}$ from the right and application of Lemma 1 with $\mu^{(-1)} = u$ gives

$$[C(m,n)]\ [u(n/m)] = [u(m/n)]\operatorname{diag}(1,2,\ldots,s),$$

which can be written as

$$\sum_{d|n} C(m,d) = \begin{cases} n & \text{if } n|m, \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

(for (2.4), see [6, p. 148]). Multiplication of (2.2) by $[u(m/n)]^{-1}$ from the left gives

$$[\mu(m/n)]\ [C(m,n)] = \operatorname{diag}(1,2,\ldots,s)\ [\mu(n/m)],$$

which can be written as

$$\sum_{d|m} \mu(m/d)C(d,n) = \begin{cases} m\,\mu(n/m) & \text{if } m|n, \\ 0 & \text{otherwise} \end{cases} \tag{2.5}$$

(for (2.5), see [3, eq. (5.4)]). Further, multiplication of (2.3) by $\big[f(n/m)\big]^{-1}$ from the right gives

$$\big[f(mn)\big]\,\big[f^{(-1)}(n/m)\big] = \big[f(m/n)\big]\operatorname{diag}\big(\mu(1)g(1), \mu(2)g(2), \ldots, \mu(s)g(s)\big),$$

which can be written as

$$\sum_{d|n} f(md)f^{(-1)}(n/d) = \begin{cases} f(m/n)\,\mu(n)\,g(n) & \text{if } n|m, \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

As (2.3) is symmetric with respect to m and n, multiplication of (2.3) from the left by $\big[f(m/n)\big]^{-1}$ gives the same identity (2.6).

2.2. Inverse forms via Dirichlet convolution

Let m be fixed. Then (1.2) can be written as

$$C(m,n) = \sum_{d|n} u(m/d)\,d\,\mu(n/d) = \big(u(m/\cdot\,)\,N * \mu\big)(n),$$

where $N(d) = d$ for all d and $*$ is the Dirichlet convolution. Multiplication by $\mu^{(-1)}$ from the right gives

$$\big(C(m,\cdot\,) * u\big)(n) = u(m/n)\,n,$$

which is equivalent to (2.4). On the other hand, let n be fixed. Then (1.2) can be written as

$$C(m,n) = \big(\mu(n/\cdot\,)\,N * u\big)(m).$$

Multiplication by $u^{(-1)}$ from the right gives

$$\big(C(\,\cdot\,,n) * \mu\big)(m) = \mu(n/m)\,m,$$

which is equivalent to (2.5).

Further, let m be fixed. Then (1.3) can be written as

$$f(mn) = \big(f(m/\cdot)\mu g * f\big)(n).$$

Multiplication by $f^{(-1)}$ from the right gives

$$\big(f(m\cdot) * f^{(-1)}\big)(n) = f(m/n)\mu(n)g(n),$$

which is equivalent to (2.6).

3. Matrix representation of type II

We now construct a matrix representation for (1.1) that in the case of (1.3) naturally gives (1.4). We first represent (1.1) in another arithmetical form, which in a sense arises from the theory of incidence functions. In fact, it can be verified that (1.1) holds for all m and n if, and only if,

$$\Psi(m/r, n/r) = \sum_{\substack{r|d|n \\ d|m}} f(d/r)g(m/d)h(n/d) \tag{3.1}$$

for all m, n and r with $r|(m,n)$. Let $\big[\Psi(m/r, n/r)\big]$ denote the $s\times s$ matrix whose n, r-entry is $\Psi(m/r, n/r)$. Then (3.1) can be written as

$$\big[\Psi(m/r, n/r)\big] = \big[g(m/r)h(n/r)\big]\,\big[f(n/r)\big]. \tag{3.2}$$

This leads to the desired inverse form as will be shown in Section 3.1.

3.1. Inverse forms via matrix algebra

If $f(1)\neq 0$, then multiplication of (3.2) by $\big[f(n/r)\big]^{-1}$ from the right gives

$$\big[\Psi(m/r, n/r)\big]\,\big[f(n/r)\big]^{-1} = \big[g(m/r)h(n/r)\big],$$

which can be written as

$$\sum_{\substack{r|d|n \\ d|m}} \Psi(m/d, n/d) f^{(-1)}(d/r) = g(m/r)h(n/r) \tag{3.3}$$

for all m, n and r with $r|(m,n)$. It can be verified that (3.3) is equivalent to

$$\sum_{d|(m,n)} \Psi(m/d, n/d) f^{(-1)}(d) = g(m)h(n) \tag{3.4}$$

for all m and n. This is the desired inverse form of (1.1).

Now, we consider the examples (1.2) and (1.3). Application of the equivalence (1.1) $\Leftrightarrow$ (3.4) shows that (1.2) holds if and only if

$$\sum_{d|(m,n)} C(m/d, n/d) d\mu(d) = \mu(n), \tag{3.5}$$

and (1.3) holds if and only if (1.4) holds.

3.2. Inverse forms via Dirichlet convolution

We here consider $\Psi(m,n)$ and $\sum_{d|(m,n)} f(d)g(m/d)h(n/d)$ as arithmetical functions of two variables m and n. The Dirichlet convolution of arithmetical functions $\alpha(m,n)$ and $\beta(m,n)$ of two variables is defined by

$$(\alpha * \beta)(m,n) = \sum_{d|m} \sum_{e|n} \alpha(d,e)\beta(m/d, n/e).$$

If f is an arithmetical function of one variable, then the principal function $P(f)(m,n)$ associated with f is defined by $P(f)(m,n) = f(n)$ if $m = n$, and $= 0$ otherwise. It is well known that $P(f * g) = P(f) * P(g)$ and $\big(P(f)\big)^{(-1)} = f^{(-1)}$, where $\big(P(f)\big)^{(-1)}$ denotes the inverse under the Dirichlet convolution of two variables [7, p. 627].

Now, we are in a position to write (1.1) in terms of the Dirichlet convolution of two variables. In fact, let $G(m,n) = g(m)h(n)$. Then (1.1) can be written as

$$\Psi(m,n) = \big(P(f) * G\big)(m,n).$$

If $f(1) \neq 0$, then multiplication by $\left(P(f)\right)^{(-1)}$ gives

$$\left(P(f^{(-1)}) * \Psi\right)(m,n) = G(m,n),$$

which can be written as (3.4).

4. A unitary analogue of the structure (1.1)

The unitary convolution of arithmetical functions f and g is

$$\sum_{d \| n} f(d) g(n/d),$$

where $d \| n$ means that $d | n$ and $(d, n/d) = 1$. Such divisor d is said to be a unitary divisor of n. A natural unitary analogue of the g.c.d. is $(m,n)_*$, which is the greatest unitary divisor of n that divides m. The unitary analogue of Ramanujan's sum has the arithmetical representation

$$C^*(m,n) = \sum_{d \| (m,n)_*} d\, \mu^*(n/d) = \sum_{\substack{d \| n \\ d | m}} d\, \mu^*(n/d), \tag{4.1}$$

where μ^* is the unitary analogue of the Möbius function. For more detailed information on unitary convolution reference is made to [5, 6].

Equation (4.1) suggests we define the unitary analogue of (1.1) by

$$\Psi^*(m,n) = \sum_{d \| (m,n)_*} f(d) g(m/d) h(n/d) = \sum_{\substack{d \| n \\ d | m}} f(d) g(m/d) h(n/d). \tag{4.2}$$

In Sections 4.1 and 4.2 we point out that the structure (4.2) possesses properties similar to those given in the above sections for the structure (1.1).

It should be noted that the natural unitary analogue of specially multiplicative functions does not coincide with the structure (4.2). Namely, it can be verified that the natural unitary analogue of the class of specially multiplicative functions is the class of multiplicative functions. This question is considered in a more general setting, for example, in [5, Exercise 4.25] and [4, Theorem 3.2.1]. We do not go into the details here.

4.1. Matrix representation of type I

Let Δ^* be defined by $\Delta^*(m,n) = 1$ if $n \| m$, and $= 0$ otherwise. Then (4.2) can be written as

$$\big[\Psi^*(m,n)\big] = \big[g(m/n)\big] \operatorname{diag}\big(f(1), f(2), \ldots, f(s)\big) \big[h(n/m)\Delta^*(n,m)\big]. \tag{4.3}$$

In particular, (4.1) can be written as

$$\big[C^*(m,n)\big] = \big[u(m/n)\big] \operatorname{diag}\big(1, 2, \ldots, s\big) \big[\mu^*(n/m)\Delta^*(n,m)\big]. \tag{4.4}$$

We next provide inverse forms for (4.4). We then need the following lemma.

LEMMA 2. If f is an arithmetical function with $f(1) \neq 0$, then

$$\big[f(m/n)\Delta^*(m,n)\big]^{-1} = \big[f^{<-1>}(m/n)\Delta^*(m,n)\big],$$

where $f^{<-1>}$ denotes the inverse function under the unitary convolution.

Proof. We have

$$\begin{aligned}
&\big[f(m/n)\Delta^*(m,n)\big]\,\big[f^{<-1>}(m/n)\Delta^*(m,n)\big] \\
&= \left[\sum_{k=1}^{s} f(m/k)\Delta^*(m,k) f^{<-1>}(k/n)\Delta^*(k,n)\right] \\
&= \left[\sum_{n\|k\|m} f(m/k)\, f^{<-1>}(k/n)\right] = \big[e_0(m/n)\big],
\end{aligned}$$

where e_0 is the identity function under the unitary convolution, that is, $e_0(m/n) = 1$ if $m = n$, and $= 0$ otherwise. Thus $\big[e_0(m/n)\big]$ is the identity matrix. This completes the proof of Lemma 2.

Now, multiplication of (4.4) by $\big[\mu^*(n/m)\Delta^*(n,m)\big]^{-1}$ from the right and application of Lemma 2 with $\mu^{*<-1>} = u$ shows that

$$\big[C^*(m,n)\big]\,\big[u(n/m)\Delta^*(n,m)\big] = \big[u(m/n)\big] \operatorname{diag}\big(1, 2, \ldots, s\big)$$

or

$$\sum_{d\|n} C^*(m,d) = \begin{cases} n & \text{if } n|m, \\ 0 & \text{otherwise} \end{cases} \tag{4.5}$$

(for (4.5), see [6, p. 188]). Further, multiplication of (4.4) by $\left[u(m/n)\right]^{-1}$ from the left and application of Lemma 1 with $u^{(-1)} = \mu$ shows that

$$\left[\mu(m/n)\right] \left[C^*(m,n)\right] = \operatorname{diag}(1,2,\ldots,s) \left[\mu^*(n/m)\Delta^*(n,m)\right]$$

or

$$\sum_{d|m} \mu(m/d)C^*(d,n) = \begin{cases} m\mu^*(n/m) & \text{if } m\|n, \\ 0 & \text{otherwise.} \end{cases} \tag{4.6}$$

The inverse forms (4.5) and (4.6) could also be derived from (4.1) with aid of the unitary convolution and the Dirichlet convolution, respectively. We do not present the details.

4.2. Matrix representation of type II

We begin by representing (4.2) in another arithmetical form. In fact, it can be verified that (4.2) holds for all m and n if, and only if,

$$\Psi^*(m/r, n/r) = \sum_{\substack{r\|d\|n \\ d|m}} f(d/r)g(m/d)h(n/d) \tag{4.7}$$

for all m, n and r with $r\|(m,n)_*$. Let $\left[\Psi(m/r,n/r)\right]$ denote the $s \times s$ matrix whose n,r-entry is $\Psi^*(m/r,n/r)$. Then (4.2) can be written as

$$\left[\Psi^*(m/r,n/r)\Delta^*(n,r)\right] = \left[g(m/r)h(n/r)\Delta^*(n,r)\right] \left[f(n/r)\Delta^*(n,r)\right]. \tag{4.8}$$

If $f(1) \neq 0$, then multiplication of (4.8) by $\left[f(n/r)\Delta^*(n,r)\right]^{-1}$ from the right gives

$$\left[\Psi^*(m/r,n/r)\Delta^*(n,r)\right] \left[f^{<-1>}(n/r)\Delta^*(n,r)\right] = \left[g(m/r)h(n/r)\Delta^*(n,r)\right],$$

which can be written as

$$\sum_{\substack{r\|d\|n \\ d|m}} \Psi^*(m/d, n/d) f^{<-1>}(d/r) = g(m/r)h(n/r) \tag{4.9}$$

for all m, n and r with $r\|(m,n)_*$. It can be verified that (4.9) is equivalent to

$$\sum_{d\|(m,n)_*} \Psi^*(m/d, n/d) f^{<-1>}(d) = g(m)h(n) \tag{4.10}$$

for all m and n.

Application of the equivalence (4.2) $\Leftrightarrow$ (4.10) to (4.1) shows that (4.1) holds if and only if

$$\sum_{d\|(m,n)_*} C^*(m/d, n/d) d\mu^*(d) = \mu^*(n). \tag{4.11}$$

The inverse form (4.10) could also be derived with aid of the 'Dirichlet-unitary' convolution of arithmetical functions of two variables defined by

$$\sum_{d|m} \sum_{e\|n} \alpha(d,e)\beta(m/d, n/e)$$

(cf. Section 3.2). We do not present the details here.

References

1. T.M Apostol, *Introduction to Analytic Number Theory,* UTM, Springer-Verlag (1976).
2. K. Bourque and S. Ligh, Matrices associated with classes of arithmetical functions, J. Number Theory 45 (1993), 367-376.
3. E. Cohen, A class of residue systems (mod r) and related arithmetical functions. I. A generalization of Möbius inversion, Pacific J. Math. 9 (1959), 13-23.
4. P. Haukkanen, Classical arithmetical identities involving a generalization of Ramanujan's sum, Ann. Acad. Fenn. Ser. A I, Math. Diss. 68 (1988), 1-69.
5. P.J. McCarthy, *Introduction to Arithmetical Functions,* Universitext, Springer-Verlag (1986).

6. R. Sivaramakrishnan, *Classical Theory of Arithmetic Functions,* Monographs and Textbooks in Pure and Applied Mathematics, Vol. 126, Marcel Dekker, Inc. (1989).

7. R. Vaidyanathaswamy, The theory of multiplicative arithmetic functions, Trans. Amer. Math. Soc. 33 (1931), 579-662.

Department of Mathematical Sciences
University of Tampere
P.O. Box 607
FIN-33101 Tampere
Finland