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GENERALIZED JACOBSTHAL REPRESENTATION SEQUENCE $\{\Upsilon_n\}$

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1. INTRODUCTION

Because of the non-Fibonacci nature of the definition of Jacobsthal numbers, various interesting aspects of these numbers merit attention *per se.* This project generalizes and extends some of the material in [4].

Consider the recurrence relation

$$\Upsilon_{n+2} = \Upsilon_{n+1} + 2\Upsilon_n + k \tag{1.1}$$

where

$$\Upsilon_0 = a, \Upsilon_1 = b; \tag{1.2}$$

a, b, k are, in general, integers. Designate this recurrence sequence by

$$\{\Upsilon_n(a, b, k)\} \tag{1.3}$$

or, simply,

$$\{\Upsilon_n\} \tag{1.3a}$$

when no confusion can exist, and, write, for later convenience,

$$c = a + b + k. \tag{1.4}$$

Also, let $J_n = \Upsilon_n(0,1,0)$. So $J_n = J_{n-1} + 2J_{n-2}$. From [4],

$$J_n = \frac{1}{3}(2^n - (-1)^n). \tag{1.5}$$

The first few members of $\{\Upsilon_n\}$ are, given by the following table.

n	0	1	2	3	4	5	6
Υ_n	a	b	$2a+b+k$	$2a+3b+2k$	$6a+5b+5k$	$10a+11b+10k$	$22a+21b+21k$

n	7	8	9	10
Υ_n	$42a+43b+42k$	$86a+85b+85k$	$170a+171b+170k$	$342a+341b+341k$

(1.6)

Induction with a little manipulation reveals that Υ_n is tied to J_n as follows:

Theorem 1:

$$\Upsilon_n = \begin{cases} cJ_n + a & n \text{ even} \quad (1.7a) \\ cJ_n - (a+k) & n \text{ odd.} \quad (1.7b) \end{cases}$$

Proof: Clearly, the Theorem is true for $n = 0, 1, 2$ $(J_0 = 0,\ J_1 = 1,\ J_2 = 1)$.

Assume the Theorem is valid for $n = 0, 1, 2, \ldots, N-1, N$.

For N even,

$$\begin{aligned} \Upsilon_{N+1} &= \Upsilon_N + 2\Upsilon_{N-1} + k && \text{by } (1.1) \\ &= \Upsilon_N - a + 2(\Upsilon_{N-1} + a + k) - (a+k) \\ &= cJ_N + 2cJ_{N-1} - (a+k) && \text{by } (1.7a),\ (1.7b) \\ &= c(J_N + 2J_{N-1}) - (a+k) \\ &= c\,J_{N+1} - (a+k) && \text{by the recurrence relation for } \{J_n\}. \end{aligned}$$

For N odd,

$$\begin{aligned} \Upsilon_{N+1} &= \Upsilon_N + 2\Upsilon_{N-1} + k && \text{by } (1.1) \\ &= \Upsilon_N + a + k + 2(\Upsilon_{N-1} - a) + a \\ &= c\,J_N + 2c\,J_{N-1} + a && \text{by } (1.7a),\ (1.7b) \\ &= cJ_{N+1} + a && \text{as before.} \end{aligned}$$

Thus, the Theorem is also valid for $N+1$ odd and $N+1$ even.

Hence, the Theorem is true.

Values of $\Upsilon_{-n} (n > 0)$ may be obtained from (1.1) by extending n through negative values. In particular,

$$\Upsilon_{-1} = \frac{1}{2}(b - a - k). \tag{1.8}$$

Without undue difficulty, we can establish by (1.2) and (1.8) the generating function

$$\begin{aligned} \sum_{i=1}^{\infty} \Upsilon_i x^{i-1} &= \frac{b + (2a - b + k)x - 2ax^2}{(1 - x - 2x^2)(1 - x)} \\ &= \frac{\Upsilon_1 + (\Upsilon_0 - \Upsilon_{-1})x - 2\Upsilon_0 x^2}{(1 - x - 2x^2)(1 - x)}. \end{aligned} \tag{1.9}$$

The generating function for the Υ_n is

$$\begin{aligned} \sum_{i=1}^{\infty} \Upsilon_{-i} x^{-i+1} &= \frac{(b - a - k) + (2a - b)x - ax^2}{(2 + x - x^2)(1 - x)} \\ &= \frac{2\Upsilon_{-1} + (2\Upsilon_0 - \Upsilon_1)x - \Upsilon_0 x^2}{(2 + x - x^2)(1 - x)}. \end{aligned} \tag{1.10}$$

Substitution of (1.5) in (1.7a) and (1.7b) produces the Binet form(s) for Υ_n.

Besides $\{J_n\}$, other sequences of interest to us are $\{j_n\}$, $\{\mathcal{J}_n\}$, and $\{\hat{j}_n\}$ for which

$$j_n = 2^n + (-1)^n \tag{1.11}$$

$$\mathcal{J}_n = \frac{1}{6}\{2^{n+3} + (-1)^n - 9\} \tag{1.12}$$

$$\hat{j}_n = \frac{1}{2}\{2^{n+2} + (-1)^n - 5\} \tag{1.13}$$

respectively. Many basic properties of these four sequences are provided in [4].

Checking the results displayed in the remaining segments of this exposition often involves the discovery of further neat facts, e.g.

$$\hat{j}_n = \begin{cases} 6\mathcal{J}_n & n \text{ even} \\ 6\mathcal{J}_n - 5 & n \text{ odd}. \end{cases} \tag{1.14}$$

It should be remarked, though perhaps it is obvious, that every fractional form in this paper does reduce to an integer, e.g. if the denominator is 3, then divisibility by 3 is always provable by elementary number-theoretic computation.

2. SPECIAL CASES

Jacobsthal-type sequences discussed in [4] are readily seen to be special cases of $\{\Upsilon_n\}$ according to the following tabulation:

Υ_n	a	b	k	c
J_n	0	1	0	1
j_n	2	1	0	3
$\mathcal{J}_n$	0	1	3	4
$\hat{j}_n$	0	1	5	6

(2.1)

Trivial cases arise when $c = 0$:

$$\Upsilon_{n(c=0)} = \begin{cases} a & n \quad \text{even} \quad (2.2a) \\ b & n \quad \text{odd} \quad (2.2b) \end{cases}$$

by (1.4) and (1.7a), (1.7b). E.g., $\Upsilon_n(1,1,-2) = 1, \Upsilon_n(1,0,-1) = \begin{cases} 1 & n \quad \text{even} \\ 0 & n \quad \text{odd}. \end{cases}$

A set of particular cases of interest is

$$\Upsilon_n(0,0,1) = \Upsilon_{n-1}(0,1,1) = \Upsilon_{n-2}(1,2,1) = \begin{cases} J_n & n \quad \text{even} \quad (2.3a) \\ J_n - 1 & n \quad \text{odd}. \quad (2.3b) \end{cases}$$

Other special instances of $\{\Upsilon_n\}$ and their values may be tabulated in grid form as follows:

$\Upsilon_n(a,b,k)$	n even	n odd
$\Upsilon_n(1,1,1)$	$3J_n+1 = 2^{2n}$	$3J_n-2=2^{2n}-1$
$\Upsilon_n(a,o,k)$	$2aJ_{n-1}+kJ_n$	$2(a+k)J_{n-1}$
$\Upsilon_n(o,b,k)$	$(b+k)J_n$	bJ_n+2kJ_{n-1}
$\Upsilon_n(a,b,o)$	$(a+b)J_n+a$	$2(a+b)J_{n-1}+b$
$\Upsilon_n(a,a,k)$	$aJ_{n+1}+kJ_n$	$aJ_{n+1}+2kJ_{n-1}$

(2.4)

In particular, $\Upsilon_n(1,2,0)=2^n$.

Subsequently, for simplicity we shall use the symbolism

$$\Upsilon_n(1,1,1) \equiv {}_1\Upsilon_n. \tag{2.5}$$

3. SOME PROPERTIES OF $\{\Upsilon_n\}$

Elementary calculations based on (1.5) and (1.7a), (1.7b) yield *inter alia*

$$\Upsilon_{n+1}+\Upsilon_n = 2^n c - k \tag{3.1}$$

$$\Upsilon_{n+r}-\Upsilon_{n-r} = 2^{n-r}\left(\frac{2^{2r}-1}{3}\right)c \tag{3.2}$$

$$3\Upsilon_{2n} = (2^{2n}-1)c+3a \tag{3.3}$$

$$3\Upsilon_{2n+1} = (2^{2n+1}+1)c-3(a+k) \tag{3.4}$$

$$\lim_{n\to\infty}\left(\frac{\Upsilon_{n+1}}{\Upsilon_n}\right) = 2 \tag{3.5}$$

$$2\Upsilon_n-\Upsilon_{n+1} = \begin{cases} 2a-b & n \text{ even} \\ -2a+b-k & n \text{ odd} \end{cases} \tag{3.6}$$

$$\Upsilon_{n+1}+\Upsilon_{n-1} = \begin{cases} \frac{2}{3}c(5.2^{n-2}-2)+2b & n \text{ even} \\ \frac{2}{3}c(5.2^{n-2}-1)+2a & n \text{ odd.} \end{cases} \tag{3.7}$$

Other identities may be deduced by applying the definitions of Υ_n and J_n. A deteminantal result concludes this short theoretical section:

$$\begin{vmatrix} \Upsilon_n & \Upsilon_{n+1} & \Upsilon_{n+2} \\ \Upsilon_{n+1} & \Upsilon_{n+2} & \Upsilon_{n+3} \\ \Upsilon_{n+2} & \Upsilon_{n+3} & \Upsilon_{n+4} \end{vmatrix} = kc(-1)^{n+1}(4a-2b+k)2^n \tag{3.8}$$

where

$$\Upsilon_n\Upsilon_{n+3}-\Upsilon_{n+1}\Upsilon_{n+2} = 2^n c\begin{cases} 2a-b & n \text{ even} \\ -2a+b-k & n \text{ odd} \end{cases} \tag{3.9}$$

has been utilized.

Verification of (3.8) and (3.9) for the special cases $\Upsilon_n = \mathcal{J}_n$ and $\Upsilon_n = \hat{j}_n$, which are supplied in [4], is worthwhile. When $\Upsilon_n = {}_1\Upsilon_n$, this case might be given cursory attention.

Simson Formula Analogues

n even:

$$\Upsilon_{n+1}\Upsilon_{n-1} - \Upsilon_n^2 = \{(b-2a-k)2^{n-1} + 2k\frac{(2^{n-2}-1)}{3}\}c + k(2a+k) \tag{3.10}$$

n odd:

$$\Upsilon_{n+1}\Upsilon_{n-1} - \Upsilon_n^2 = \{(-b+2a-k)2^{n-1} + 2k\frac{(2^n+1)}{3}\}c - k(2a+k). \tag{3.11}$$

While these forms necessarily appear somewhat like a mathematical ugly duckling, in special cases – cf.(2.1) and [4] - they can be very fine swans indeed!

Furthermore, (3.10), (3.11), and (2.5) lead to

$$_1\Upsilon_{n+1}\, {}_1\Upsilon_{n-1} - {}_1\Upsilon_n^2 = \begin{cases} -5.2^{n-1}+1 & n \text{ even} \\ 2^{n+1}-1 & n \text{ odd.} \end{cases} \tag{3.12}$$

Associated Sequences

Define

$$\Upsilon_n^{(k)} = \Upsilon_{n+1}^{(k-1)} + 2\Upsilon_{n-1}^{(k-1)}, \tag{3.13}$$

where $\Upsilon_n^{(0)} \equiv \Upsilon_n$, to be the k^{th} *associated sequence* of $\{\Upsilon_n\}$. See [2] for other developments of this concept.

Now

$$\Upsilon_n^{(1)} = \Upsilon_{n+1} + 2\Upsilon_{n-1} = \begin{cases} cj_n - 3(a+k) & n \text{ even} \\ cj_n + 3a & n \text{ odd,} \end{cases} \tag{3.14}$$

$$\Upsilon_n^{(2)} = \Upsilon_{n+1}^{(1)} + 2\Upsilon_{n-1}^{2} = \begin{cases} 9cJ_n + 9a & n \text{ even} \\ 9cJ_n - 9(a+k) & n \text{ odd,} \end{cases} \tag{3.15}$$

and so on. Eventually

$$\Upsilon_n^{(2m)} = 3^{2m}\begin{cases} cJ_n + a & n \text{ even} \\ cJ_n - (a+k) & n \text{ odd,} \end{cases} \tag{3.16}$$

$$\Upsilon_n^{(2m+1)} = 3^{2m}\begin{cases} cj_n - 3(a+k) & n \text{ even} \\ cj_n + 3a & n \text{ odd.} \end{cases} \tag{3.17}$$

Examples:

(i) $\Upsilon_n = J_n(a=0, k=0, c=1)$:

$$J_n^{(2m)} = 3^{2m} J_n \tag{3.18}$$

$$J_n^{(2m+1)} = 3^{2m} j_n. \tag{3.19}$$

(ii) $\Upsilon_n = j_n (a = 2,\ k = 0,\ c = 3)$:

$$j_n^{2m} = 3^{2m} j_n \tag{3.20}$$

$$j_n^{(2m-1)} = 3^{2m} J_n. \tag{3.21}$$

(iii) $\Upsilon_n = \mathcal{J}_n (a = 0,\ k = 3,\ c = 4)$:

$$\mathcal{J}_n^{(2m)} = 3^{2m} \mathcal{J}_n \tag{3.22}$$

$$\mathcal{J}_n^{2m+1)} = 3^{2m} (\hat{j}_{n+1} - 2). \tag{3.23}$$

(iv) $\Upsilon_n = \hat{j}_n (a = 0,\ k = 5,\ c = 6)$:

$$\hat{j}_n^{(2m)} = 3^{2m} \hat{j}_n \tag{3.24}$$

$$\hat{j}_n^{(2m+1)} = 3^{2m+1} (3\mathcal{J}_{n-1} + 2). \tag{3.25}$$

(v) $\Upsilon_n = {}_1\Upsilon_n (a = 1,\ k = 1,\ c = 3)$:

$${}_1\Upsilon_n^{(2m)} = \begin{cases} 3^{2m} 2^n & n \text{ even} \\ 3^{2m} j_n & n \text{ odd} \end{cases} \tag{3.26}$$

$${}_1\Upsilon_n^{(2m+1)} = \begin{cases} 3^{2m+2} J_n & n \text{ even} \\ 3^{2m+1} 2^n & n \text{ odd.} \end{cases} \tag{3.27}$$

See also [4] and [5].

4. CONCLUSION

There are at least two directions in which this outline of the features of $\{\Upsilon_n\}$ could be developed:

(i) extension of the theory to negative subscripts, i.e., $\{\Upsilon_{-n}\}$, $n > o$, and

(ii) generalization of the number-theoretic properties of $\{\Upsilon_n\}$ to properties of polynomial sequences $\{\Upsilon_n(x)\}$.

Progress with both (i) and (ii) has been made.

One might also consider a possible analysis of the plane curves aspect of our sequences. Consult [1] for some ideas.

Reference [3] contains material which, for Pell numbers, in a sense complements some of the theory developed in this presentation.

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