NNTDM 2 (1996) 3, 3-4

A GENERALIZATION OF ONE ERDÖS'S PROBLEM

Stephan Danchev[1], Krassimir Atanassov[1,2]

[1] *Centre for Biomedical Engineering - Bulg. Academy of Sci. Acad. G. Bonchev str., Bl. 105, Sofia-1113, Bulgaria e-mail: krat@bgcict.bitnet*
[2] *Math. Res. Lab., P.O.Box 12, Sofia-1113, Bulgaria e-mail: krat@bgcict.bitnet*

In memory of
Prof. Paul Erdös

The world-famous Hungarian mathematician Paul Erdös formulated in the Bulgarian newspaper "Mathematical Post" No. 11/June 1993 the following problem: let the natural numbers $a_1, a_2, ..., a_{n+1}$ satisfy the inequalities:

$$1 \leq a_1 \leq a_2 \leq ... \leq a_{n+1} \leq 2.n.$$

Prove that there exist two numbers among them such that one of them divides the other.

We shall formulate a generalization of the above Erdös's problem.

Let $a_1, a_2, ..., a_n$ are integers in the interval $[1, m]$, where $2.\lceil \frac{m}{2} \rceil \geq n > \lceil \frac{m}{2} \rceil$. The following assertion is valid.

<u>THEOREM 1</u>: There exist at least $n - \lceil \frac{m}{2} \rceil$ pairs unordered and without reiteration) of these numbers, such that one of the numbers divides the other and this estimation is exact.

<u>Proof</u>: For two fixed natural numbers m and n, let S_2 denote the number of the pairs which satisfy the condition that the result of the division is a power of 2. We shall prove that $S_2 \geq n - \lceil \frac{m}{2} \rceil$. Let

$$A_p = \{a_i | \ a_i = 2^q.(2p-1), \ 1 \leq i \leq n, \ p, q \in \mathcal{N}\}$$

be a multiset, where $\mathcal{N}$ is the set of the natural numbers, $1 \leq p \leq \lceil \frac{m}{2} \rceil$. Obviously,

$$S_2 = \sum_{p=1}^{\lceil \frac{m}{2} \rceil} \binom{\overline{A_p}}{2},$$

where $\overline{X}$ is the cardinality of the set X.

We shall prove that the minimum of S_2 is obtained for some A_p, $(1 \leq p \leq \lceil \frac{m}{2} \rceil)$ that satisfy $| \ \overline{A_s} - \overline{A_r} \ | \leq 1$, for all s and r such that $1 \leq s < r \leq \lceil \frac{m}{2} \rceil$.

Let us assume the opposite, i.e., that there exist s and r for which $\overline{A_s} - \overline{A_r} > 1$.

Let us transfer one element of the set A_s to the set A_r. Therefore, we obtain the new sets A'_s and A'_r and the new number

$$S'_2 = \sum_{p=1}^{\lceil \frac{m}{2} \rceil} \binom{\overline{A}'_p}{2}.$$

Then

$$S_2' - S_2 = \binom{\overline{A}_s - 1}{2} + \binom{\overline{A}_r + 1}{2} - \binom{\overline{A}_s}{2} - \binom{\overline{A}_r}{2} = \overline{A}_r - \overline{A}_s + 1 < 0,$$

which is a contradiction with our assumption of the minimality of S_2.

When $2.\lceil\frac{m}{2}\rceil \geq n > \lceil\frac{m}{2}\rceil$ it is easily seen that the minimal value of S_2 is obtained for $\overline{A}_p = 2$ for $n - \lceil\frac{m}{2}\rceil$ different values of p and $\overline{A}_p = 1$ for the rest values of p. This minimum is equal to $n - \lceil\frac{m}{2}\rceil$. The lower estimation is exact, which follows from the existence of the multiset $\{m, m-1, ..., m-(\lceil\frac{m}{2}\rceil - 1), m, m-1, ..., m-(n-\lceil\frac{m}{2}\rceil - 1)\}$. In this multiset the pairs for which one number divides the other are exactly $n - \lceil\frac{m}{2}\rceil$. ◇

Let $a_1, a_2, ..., a_n$ be independently chosen by an uniform discrete distribution in the interval $[1, m]$. We shall prove the following assertion.

<u>THEOREM 2</u>: The mean value of the number of the pairs (unordered and without reiterations) os these numbers, such that one of them is a divisor of the other, is $\binom{n}{2}\frac{2\ln m + O(1)}{m}$.
<u>PROOF</u>: Let P be the probability that from two arbitrary numbers in the interval $[1, m]$ one of them divides the other; let the mean value be $E(n)$. Because the $n-$th number is chosen indepedently from the first $n-1$ numbers, then:

$$E(n) = E(n-1) + (n-1).P$$

From $E(2) = P$ it follows that $E(n) = \binom{n}{2}.P$.

Let a and b be the chosen numbers. Then P is equal to

$$\frac{\overline{\{(a,b) : 1 \leq a = b \leq m\}} + \overline{\{(a,b) : a|b,\ 1 \leq a < b \leq m\}} + \overline{\{(a,b) : b|a,\ 1 \leq b < a \leq m\}}}{m^2} =$$

$$= \frac{m + \sum_{a=1}^{m-1}(\lfloor\frac{m}{a}\rfloor - 1) + \sum_{b=1}^{m-1}(\lfloor\frac{m}{b}\rfloor - 1)}{m^2} = \frac{2\sum_{k=1}^{m}\lfloor\frac{m}{k}\rfloor - m}{m^2}.$$

From

$$\sum_{k=1}^{m}\frac{m}{k} \geq \sum_{k=1}^{m}\lfloor\frac{m}{k}\rfloor \geq \sum_{k=1}^{m}(\frac{m}{k} - 1) = \sum_{k=1}^{m}\frac{m}{k} - m$$

and

$$\sum_{k=1}^{m}\frac{1}{k} = \ln m + O(1)$$

it follows that

$$\sum_{k=1}^{m}\lfloor\frac{m}{k}\rfloor = m\ln m + O(m).$$

Then $P = \frac{2\ln m + O(1)}{m}$ and $E(n) = \binom{n}{2}\frac{2\ln m + O(1)}{m}$. ◇