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A MODIFICATION OF ONE AVANESSOV'S PROBLEM

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The following open problem of E. Avanessov is introduced in [1]: to find all natural numbers x for which there exist natural numbers n, such that

$$x = n + \varphi(n),$$

where function φ is defined below (see [2,3]).

Using the definition of function ψ we can formulate the following similar to the above problem: to find all natural numbers x for which there exist natural numbers n, such that

$$x = n + \psi(n), \tag{1}$$

where function ψ is defined below, too (see [2,3]).

For

$$n = \sum_{i=1}^{m} a_i . 10^{m-i} \equiv a_1 a_2 \ldots a_m,$$

where a_i is a natural number and $0 \le a_i \le 9$ $(1 \le i \le m)$ let (see [2,3]):

$$\varphi(n) = \begin{cases} 0 & , \text{ if } n = 0 \\ \sum_{i=1}^{m} a_i, & \text{otherwise} \end{cases}$$

and for the sequence of functions $\varphi_0, \varphi_1, \varphi_2, \ldots,$ where (l is natural number)

$$\varphi_0(n) = n, \quad \varphi_{l+1}(n) = \varphi(\varphi_l(n)),$$

let the function ψ be defined by $\psi(n) = \varphi_l(n)$, in which

$$\varphi_{l+1}(n) = \varphi_l(n).$$

Now we shall prove the following

THEOREM: For all natural number $x \ge 10$ there exists an unique natural number n, such that (1) is valid.

Proof: Let $x = 18.k + 2.r$, where k, r are natural numbers and $0 \le r \le 8$. Then for the natural number $n = x - r$:

$$\begin{aligned} x - r + \psi(x - r) &= 18.k + r + \psi(18.k + r) \\ &= 18.k + r + r \\ &= 18.k + 2.r \\ &= x. \end{aligned}$$

Let $x = 18.k + 2.r + 9$, where k, r are natural numbers and $1 \le r \le 9$. Then for the natural number $n = x - r$:

$$x - r + \psi(x - r) = 18.k + r + 9 + \psi(18.k + r + 9)$$
$$= 18.k + 2.r + 9$$
$$= x.$$

Therefore, for every natural number x there exists at least one solution of (1). Now, we shall prove the uniqueness.

Let

$$x = m + \psi(m) = n + \psi(n), \tag{2}$$

i.e. let x has at least two different representations in the form of (1). Then

$$m - n = \psi(n) - \psi(m).$$

Let $m > n$. from $\psi(n) - \psi(m) \le 8$ it follows that, if $m = n + k$, then $1 \le k \le 8$.

Therefore,

$$m + \psi(m) = n + k + \psi(n + k) = n + \psi(n),$$

i.e.

$$\psi(n + k) = \psi(n) - k. \tag{3}$$

Let $g(n, k) = \psi(n + k) - \psi(n) + k$.

It can be seen directly that

$$g(n + 9.s, k) = \psi(n + 9.s, k) - \psi(n + 9.s) - k$$
$$= \psi(n, k) - \psi(n) - k.$$

Therefore, we can consider only the values of $g(n, k)$ for $0 \le n \le 9$. These are the following:

n \ k	1	2	3	4	5	6	7	8
0	2	4	6	8	10	12	14	16
1	2	4	6	8	10	12	14	16
2	2	4	6	8	10	12	14	7
3	2	4	6	8	10	12	5	7
4	2	4	6	8	10	3	5	7
5	2	4	6	8	1	3	5	7
6	2	4	6	-1	1	3	5	7
7	2	4	-3	-1	1	3	5	7
8	2	-5	-3	-1	1	3	5	7
9	-7	-5	-3	-1	1	3	5	7

Therefore, for every two natural numbers n, k ($1 \le k \le 8$):

$$g(n, k) \ne 0,$$

i.e., (3) has no solution. Therefore the representation (1) of x is unique.

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