NNTDM 2 (1996), 2, 1-26

THE GOLDBACH PROBLEM (II) (Continuation)

by Aldo Peretti

11. We now pass to calculate $\mathcal{L}^2\{\vartheta(x)\}$. From [79] we obtain:

[81] $$\mathcal{L}^2\{\vartheta(x)\} = \frac{F^2(n)}{s^2(1-e^{-s})^2} + \frac{A^2}{s^2} n^{2\vartheta+1/2+\varepsilon} +$$

$$+ \sum_{q=1}^{[\sqrt{n}]} \sum_{h=1}^{q-1} \frac{\mu^2(q)}{\varphi^2(q)\ s^2 (s+2\pi ih/q)^2} + \frac{2AF(n)\, n^{\vartheta+1/4+\varepsilon}}{s^2(1-e^{-s})} +$$

$$+ \sum_{q=1}^{[\sqrt{n}]} \sum_{h=1}^{q-1} \frac{2F(n)\ \mu(q)}{s^2(1-e^{-s})(s+2\pi ih/q)\,\varphi(q)} + \sum_{q=1}^{[\sqrt{n}]} \sum_{h=1}^{q-1} \frac{2A\, n^{\vartheta+1/4+\varepsilon}\ \mu(q)}{\varphi(q)\ s^2(s+2\pi ih/q)} +$$

$$+ \sum_{\substack{q_1=1 \\ q_1\neq q_2}}^{[\sqrt{n}]} \sum_{\substack{h_1 \\ h_1\neq h_2}}^{q-1} \sum_{q_2=1}^{[\sqrt{n}]} \sum_{h_2=1}^{q-1} \frac{\mu(q_1)\ \mu(q_2)}{\varphi(q_1)\ \varphi(q_2)\ s^2 (s+2\pi ih_1/q_1)\ (s+2\pi ih_2/q_2)}$$

12. Accordingly to [61], we must evaluate either:

[82] $$\mathcal{L}^{-1}\{(1-e^{-s})^2\ \mathcal{L}^2\ \{\vartheta(x)\}$$

or, alternatively:

[83] $$\Delta^{2)}\ \mathcal{L}^{-1}\{\ \mathcal{L}^2(\vartheta(x))\}$$

We apply [82] to the terms of [81] that contain $1-e^{-s}$ in the denominator. To the remainder ones we apply [83].

Thus [81] turns out to be:

[84] $$v^*(t) = \frac{v(t+o) + v(t-o)}{2} = F(n)^2 \mathcal{L}^{-1}(1/s^2) + \Delta^{2)} n^{2\vartheta+\frac{1}{2}+\varepsilon} \mathcal{L}^{-1}(1/s^2) +$$

$$+ \Delta^{2)} \sum_q \sum_h \frac{\mu^2(q)}{\varphi^2(q)} \mathcal{L}^{-1}\left\{\frac{1}{s^2(s+2\pi ih/q)^2}\right\} + 2AF(n)\, n^{\vartheta+1/4+\varepsilon} \mathcal{L}^{-1}\left\{\frac{1-e^{-s}}{s^2}\right\}$$

$$+ \sum_q \sum_h \frac{2F(n)\,\mu(q)}{\varphi(q)} \mathcal{L}^{-1}\left\{\frac{1-e^{-s}}{s^2(s+2\pi ih/q)}\right\} + \sum_q \sum_h \frac{2A\, n^{\vartheta+1/4+\varepsilon}}{\varphi(q)} \Delta^{2)} \mathcal{L}^{-1}\left\{\frac{1}{s^2(s+2\pi ih/q)}\right\}$$

$$+ \Delta^{2)} \sum_{q_1} \sum_{h_1} \sum_{q_2} \sum_{h_2} \frac{\mu(q_1)\,\mu(q_2)}{\varphi(q_1)\,\varphi(q_2)} \mathcal{L}^{-1}\left\{\frac{1}{s^2(s+2\pi ih_1/q_1)(s+2\pi ih_2/q_2)}\right\}$$

the limits in the sums being the same than before.

13. For the inverse transforms we have the following elementary evaluations ($A=2\pi ih/q$).

a) $$F_1(t) = \mathcal{L}^{-1}(1/s^2) = t$$

b) $$F_2(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s+A)^2}\right\} = \int_0^t (t-u)\ ue^{-At} =$$

$$= \frac{t}{A^2}(1+e^{-At}) + \frac{2e^{-At}}{A^3} - \frac{2}{A^3}$$

c) $$F_3(t) = \mathcal{L}^{-1}\left\{\frac{1-e^{-s}}{s^2}\right\} = \begin{cases} 1 & \text{if } t > 1 \\ t & \text{if } t < 1 \end{cases}$$

d) $$F_4(t) = \mathcal{L}^{-1}\left\{\frac{1-e^{-s}}{s^2(s+A)}\right\} = \int_0^t (t-u)\ e^{-Au} - \int_0^{t-1} (t-u-1)\ e^{-Au}\ du =$$

$$= \frac{1}{A} + \frac{e^{-At}}{A^2} (1-e^{A})$$

e) $$F_5(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2 (s+A)}\right\} = \int_o^t (t-u)\ e^{-Au}\ du_o = \frac{t}{A} + \frac{e^{-At}}{A^2} - \frac{1}{A^2}$$

f) $$F_6(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2 (s+A_1)(s+A_2)}\right\} = \frac{1}{(A_2-A_1)}\left\{\frac{1}{A_1}\left(t - \frac{1-e^{-A_1 t}}{A_1}\right) - \frac{1}{A_2}\left(t - \frac{1-e^{-A_2 t}}{A_2}\right)\right\}$$

From here can be deduced the following results:

g) $$\Delta^{2)}\ F_1(t) = o$$

h) $$\Delta^{2)}\ F_2(t) = \Delta^{2)} \frac{t}{A^2} e^{-At} + \frac{2}{A^3} \Delta^{2)}\ e^{-At}$$

i) $$\Delta^{2)}\ F_5(t) = \frac{1}{A^2} \Delta^{2)}\ e^{-At}$$

j) $$\Delta^{2)}\ F_6(t) = \frac{1}{(A_2-A_1)}\left\{\frac{\Delta^{2)}\ e^{-A_1 t}}{A_1} - \frac{\Delta^{2)}\ e^{-A_2 t}}{A_2}\right\}$$

Hence, formula [84] can be written as:

[85] $$\frac{\nu(t+o) + \nu(t-o)}{2} = F(n)^2\, t + \sum_q \left(\frac{\mu(q)}{\varphi(q)}\right)^2 \sum_h \Delta^{2)}\ F_2(t) + 2AF(n)\, n^{\vartheta+1/4+\varepsilon} +$$

$$+ 2F(n) \sum_q \frac{\mu(q)}{\varphi(q)} \sum_h F_4(t) + 2A\, n^{\vartheta+1/4+\varepsilon} \sum_q \frac{1}{\varphi(q)} \sum_h \Delta^{2)}\ F_5(t)$$

$$+ \sum_{q_1} \sum_{h_1} \sum_{q_2} \sum_{h_2} \frac{\mu_1(q_1)\ \mu(q_2)}{\varphi(q_1)\ \varphi(q_2)}\ \Delta^{2)'}\ F_6(t) =$$

[86]
$$= F(n)^2\, t + \sum_q \left(\frac{\mu(q)}{\varphi(q)}\right)^2 \sum_h \left\{\Delta^{2)}\ \frac{t}{A^2}\ e^{-At} + \frac{2}{A^3}\ \Delta^{2)}\ e^{-At}\right\}$$

$$+ 2AF(n)\, n^{\vartheta+1/4+\varepsilon} + 2F(n) \sum_q \frac{\mu(q)}{\varphi(q)} \sum_h \left\{\frac{1}{A} + \frac{e^{-At}}{A^2}\ (1-e^{A})\right\}$$

$$+ 2An^{\vartheta+1/4+\varepsilon} \sum_q \frac{1}{\varphi(q)} \sum_h \Delta^{2)}\ \frac{e^{-At}}{A^2} +$$

$$+ \sum_{q_1} \sum_{h_1} \sum_{q_2} \sum_{h_2} \frac{\mu(q_1)\ \mu(q_2)}{\varphi(q_1)\ \varphi(q_2)}\ \Delta^{2)}\ F_6(t)$$

14. At this stage of the reasoning, we must make the same remark that was stated in paragraph 5: we have treated the variable t as if it were continuous. But this is not the case as t is always a natural number. Hence, the inverse transforms that appear in paragraph 13 should be written as the average at the right and left of t .

For instance, in change of formula 13 a) we should write:

$$F_1(t) = \mathcal{L}^{-1}\ (1/s^2) = \frac{(t+o) + (t-o)}{2}$$

and so on with all the subsequent formulas.

We can then equate all the terms in (t + o) and (t − o) that should appear at both sides of [86] in order to obtain:

[87]
$$\nu(t) = F(n)^2\, t + \sum_q \left(\frac{\mu(q)}{\varphi(q)}\right)^2 \sum_h \Delta^{2)}\ F_2(t) + 2AF(n)\, n^{\vartheta+1/4+\varepsilon} +$$

$$+\ 2F(n) \sum_{q} \frac{\mu(q)}{\varphi(q)} \sum_{h} F_4(t) \ +\ 2A\, n^{\vartheta+1/4+\varepsilon} \sum_{q} \frac{1}{\varphi(q)} \sum_{h} \Delta^{2)}\ F_5(t)\ +$$

$$+ \sum_{q_1} \sum_{h_1} \sum_{q_2} \sum_{h_2} \frac{\mu(q_1)\ \mu(q_2)}{\varphi(q_1)\ \varphi(q_2)}\ \Delta^{2)}\ F_6(t)$$

It is clear now the order that we must follow in order to evaluate $\nu(t)$:

- In first place we must calculate $\Delta^{2)}\ e^{-At}$ and $\Delta^{2)}\ te^{-At}$ (we shall see that the second case can be reduced to the first).

- In second place we must evaluate the sums that run along h.

- In third place we must evaluate the sums that run along q.

- $F(n)$ and the sum in the last line will be evaluated apart.

15. Calculation of $\Delta^{2)}\ e^{-At}$

For $\Delta^{n)}\ e^{-At}$ we can make use of the formula (6).

[88] $$\Delta^{n)} \begin{matrix}\sin\\ \cos\end{matrix} (Bt) = \left(2 \sin \frac{B}{2}\right)^2 \begin{matrix}\sin\\ \cos\end{matrix} \left\{Bt + n\, \frac{(B+\pi)}{2}\right\}$$

with $n = 2$. As

$$e^{-At} = \cos|A|t - i \sin|A|t \qquad (A = 2\pi i h/q)$$

then

[89] $$\Delta^{2)}\ e^{-At} = \Delta^{2)} \cos|A|t - i\ \Delta^{2)} \sin|A|t$$

$$= \left(2 \sin \frac{|A|}{2}\right)^2 \left\{\cos(|A|t + |A| + \pi) - i \sin(|A|t + |A| + \pi)\right\}$$

$$= - \left(2 \sin \frac{|A|}{2} \right)^2 e^{-A(t+1)}$$

However in [106] we shall find a more suitable result.

16. Calculation of $\Delta^{2)}\, te^{-At}$.

The n - th differences of a function can be expressed in terms of its derivatives by means of the formula (6).

[90] $$\Delta^{n)} u = \frac{d^n u}{dt^n} + \frac{\Delta^{n)}\, 0^{n+1}}{(n+1)\,!} \frac{d^{n+1} u}{dt^{n+1}} + \frac{\Delta^{n)}\, 0^{n+2}}{(n+2)!} \frac{d^{n+2} u}{dt^{n+2}} + \ldots$$

where

[91] $$\Delta^{n)}\, 0^m = n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \binom{n}{3}(n-3)^m + \ldots$$

But according to N. Nielsen (7) we have that

[92] $$\Delta^{n)}\, 0^m = n! \binom{m}{n} C_{n+1}^{m-n}$$

where C^r_{n+1} are the Stirling numbers of the second kind, that can be expressed through the Stirling polynomials $\Psi_r(x)$ by means of the formula

[93] $$C^r_{n+1} = \frac{(-1)^{r+1}\,(n+r)!}{(n-1)!} \Psi_{r-1}(-n-1)$$

Hence

[94] $$\Delta^{n)}\, 0^m = (-1)^{m-n-1} \frac{(m!)^2}{(m-n)!\,(n-1)!} \Psi_{m-n-1}(-n-1)$$

The generating function of the Stirling polynomials is:

$$[95] \qquad \left(\frac{\alpha}{1-e^{-\alpha}}\right)^{x+1} = 1 + (x+1) \sum_{r=o}^{\infty} \Psi_r(x)\, \alpha^{r+1} \qquad (|\alpha| < 2\,\pi)$$

Replacing the value of [92] in [90] we get:

$$[96] \qquad \Delta^{n)} u = \frac{d^n}{dt^n} u + \frac{n(n+1)}{1!} \Psi o(-n-1) \frac{d^{n+1}}{dt^{n+1}} - \frac{n(n+1)\,(n+2)}{2!} \Psi_1(-n-1) \frac{d^{n+2}u}{dt^{n+2}} + \dots$$

This formula seems to be new.

Comparing [96] with [95] we deduce that development [96] is valid if

$$[97] \qquad \left| \binom{n+r}{r} \frac{d^{n+r}u}{dt^{n+r}} \right| < (2\pi)^{r+c} \; F(t) \qquad \text{(c: absolute constant)} \quad \text{(F(t): arbitrary function)}$$

In our case we are concerned with second differences, hence n=2 and [96] takes the particular form:

$$[98] \qquad \Delta^{2)} u = \frac{d^2u}{dt^2} + 2.3\, \Psi_o(-3) \frac{d^3u}{dt^3} - 3.4.\, \Psi_1(-3) \frac{d^4u}{dt^4} + \dots$$

The series at right converges if

$$[99] \qquad \left| (r+1)\;(r+2) \frac{d^{r+2}u}{dt^{r+2}} \right| < (2\pi)^{r+c} \;\; \varphi(t)$$

for every r sufficiently large.

It can be checked easily by induction that $d^{r+2}u / dt^{r+2}$ in the case in which $u = u(t) = te^{-At}$ is a sum of r+2 terms, in which the dominant one is $A^{r+2}\, te^{-At}$, and as $|A| < 2\,\pi$, condition [97] is fulfilled and development [98] is convergent.

But the formula so obtained is excessively complicated, so that we choose an alternative way in paragraph 18.

17. Spite its failure in the evaluation of $\Delta^{2)}\, te^{-At}$, formula [98] is useful in order to calculate $\Delta^{2)}\, t^{2+\rho}$, that appeared before in paragraph 5.
There we had:

[100] $$\Delta^{2)}\ t^{2+\rho} \ = \ t^{2+\beta}\ \Delta^{2)}\ t^{i\gamma}\ \{1+0\ (1/t)\}$$

Hence we are led to evaluate $\Delta^{2)}\ t^{i\gamma}$. But:

$$\frac{d^2}{dt^2}\ t^{i\gamma} = -\frac{\gamma^2}{t^2}\ t^{i\gamma} - \frac{i\gamma}{t^2}\ t^{i\gamma}$$

while in the higher derivatives the dominant term is $0\ (\gamma^r / t^r)$.
According to [98] is:

[101] $$\Delta^{2)}\ t^{i\gamma} = -\frac{\gamma^2}{t^2}\ t^{i\gamma}\ \{1+0\ (1)\}$$

as far as $|\gamma| < 2\,\pi\, t$.
From [101] follows finally:

[102] $$\Delta^{2)}\ t^{2+\rho} = -\ \gamma^2\ t^{\beta}\ t^{i\gamma}\ \{1+0\ (1)\}$$

[103] $$|\Delta^{2)}\ t^{2+\rho}| \ < \ c_1\ \gamma^2\ t^{\vartheta} \qquad (|\gamma| < 2\,\pi\, t)$$

In the same fashion can be proved that:

[104] $$\Delta^{2)}\ t^{\rho_1+\rho_2-1} = c_1\,(\gamma_1+\gamma_2)^2\ t^{\beta_1+\beta_2-1}\ t^{i(\gamma_1+\gamma_2)}$$

i.e.:

$$|\Delta^{2)}\ t^{\rho_1+\rho_2-1}| \ \leq\ c_2\,(\gamma_1+\gamma_2)^2\ t^{2\vartheta-1}$$

if $|\gamma_1 + \gamma_2| < c_3 t$, as stated in paragraph 7.

18. We return now to the alternative evaluation of $\Delta^{2)} te^{-At}$.
We have:

$$[105] \quad \Delta^{2)} te^{-At} = te^{-At} - 2(t-1)\ e^{-A(t-1)} + (t-2)\ e^{-A(t-2)} =$$

$$= t\ \{e^{-At} - 2e^{-A(t-1)} + e^{-A(t-2)}\}\ \{1+0\,(1/t)\} =$$

$$= t\ \Delta^{2)}\ e^{-At}\ \{1+0\,(1/t)\}$$

Leaving aside the exact expression given by [96] for $\Delta^{2)}\ te^{-At}$ we employ it now with $u = u(t) = e^{-At}$, and obtain the new formula.

$$[106] \quad \Delta^{2)} e^{-At} = A^2 e^{-At} + 2.3\,\Psi o\, A^3\ e^{-At} - 3.4\,\Psi_1(-3)\, A^4\ e^{-At} + \dots$$

$$= A^2 e^{-At} \{1 + 2.3\,\Psi o\, A - 3.4\,\Psi_1(-3)\, A^2 + \dots\} = A^2 e^{-At} M$$

The right hand series converges because $|A| < 2\pi$. Denote by M its sum. Then, by [105] and [106] we have:

$$[107] \qquad \Delta^{2)}\ e^{-At} = t\ A^2\ e^{-At}\ M\ \{1+0\,(1/t)\}$$

Obviously, $M \le M_0$ = absolute constant.
We are now in position to evaluate $\Delta^{2)}\ F_2(t)$; $\Delta^{2)}\ F_4(t)$; $\Delta^{2)}\ F_5(t)$ and $\Delta^{2)}\ F_6(t)$.

19. Before performing the effective calculus of these $\Delta^{2)}$, we introduce here a new arithmetic function that will make our formulas more compact and simple.
Ramanujan sums $C_q(n)$ are defined for n = integer number by:

[108] $$C_q(n) = \sum_h e^{2\pi i h n/q} = \sum_h \cos 2\pi h n/q$$

where h runs through all the values relatively prime with q , and less than q , h=o being included when q=o , but not otherwise (PN III, p. 26).
We can define then $C_q(n)$ as the sum of the $\varphi(q)$ primitive roots of unity of order q, risen (each one) to the n-th power.
We have, obviously, that:

$$C_q(-n) = C_q(n)$$

Ramanujan considered only the case where n is an integer.
Let us now define as "Ramanujan function" to the same sum:

[109] $$C_q(t) = \sum_h e^{-2\pi i h t/q}$$

that only differs from [108] in the fact that t is a continuous variable.
This function is intimately tied to our problem, because

[110] $$\mathcal{L}\ \{C_q(t)\} = \sum_{\substack{h=o \\ (h,q)=1}}^{q-1} \frac{1}{s+2-\pi i h/q}$$

From here follows, by [75]

[111] $$\mathcal{L}\ \{\vartheta(x)\} = \sum_p \frac{1}{s} \log p\ e^{-ps} = \frac{1}{s} \sum_{q=1}^{N} \frac{\mu(q)}{\varphi(q)} \mathcal{L}\ \{C_q(t)\} + \frac{R_N}{s}$$

where R_N is the error performed when we make the Farey dissection of order N.

20. Once calculated $\Delta^{2)}\ e^{-At}$ and $\Delta^{2)}\ te^{-At}$ through [107] and [105], the next thing we must compute in [87] are the sums extended along h, i.e.:

[112] $$\sum_{\substack{h=1\\(h,q)=1}}^{q-1} \frac{\Delta^{2)}\, e^{-At}}{A^2} \qquad\qquad \sum_{\substack{h=1\\(h,q)=1}}^{q-1} \frac{\Delta^{2)}\, e^{-At}}{A^3}$$

[113] and $$\sum_{\substack{h=1\\(h,q)=1}}^{q-1} \frac{\Delta^{2)}\, te^{-At}}{A^2}$$

From [106] follows:

[114] $$\sum_{\substack{h=1\\(h,q)=1}}^{q-1} \frac{\Delta^{2)}\, e^{-At}}{A^2} = \sum_{\substack{h=1\\(h,q)=1}}^{q-1} e^{-2\pi ih/q} \left\{1 + 2.3\, \Psi_o\,(2\pi ih/q) + 3.4\, \Psi_1(-3)\, 4\pi^2 h^2/q^2 + \ldots\right\}$$

$$= \sum_h e^{-2\pi ih/q} + 2.3\,\frac{2\Psi_o\pi i}{q} \sum_h h\, e^{-2\pi ih/q} - 3.4\, \frac{2^2\pi^2\Psi_1(-3)}{q^2} \sum_h h^2\, e^{-2\pi ih/q} - \ldots$$

[115] $$= C_q(t) + 2.3\, \frac{2\Psi_o\pi i}{q}\, C'_q(t) - 3.4\, \frac{2^2\pi^2\Psi_1(-3)}{q^2}\, C''_q(t) + \ldots$$

The convergence of this development follows from the fact that [115] is merely the sum of a finite number of series of the type [103], that were known to be convergent because $|2\pi ih/q| < 2\pi$. In order to evaluate $C^{m)}_q(t)$ we have not at present anything better than theorem 328 of ref. (5):

[116] $$\left| \sum_{\substack{h=1\\(h,q)=1}}^{q-1} h^m\, e^{-2\pi iht/q} \right| < C_o q^{m+\varepsilon} \qquad \varepsilon > 0 \qquad m \geq 1$$

Hence,

$$C^{m)}(t) = 0\,(q^{m+\varepsilon})$$

It follows, replacing this bound in [115], that

[117] $$\sum_{\substack{h=1\\(h,q)=1}}^{q-1} \Delta^{2)} \frac{e^{-At}}{A^2} = 0\,(q^{\varepsilon}) + C_q(t)$$

21. We analyze now the case of

$$\sum_h \Delta^{2)} \frac{e^{-At}}{A^3}$$

Then [114] changes to:

[118] $$\sum_{\substack{h=1\\(h,q)=1}}^{q-1} \Delta^{2)} \frac{e^{-At}}{A^3} = \sum_{\substack{h=1\\(h,q)=1}}^{q-1} \frac{q}{2\pi ih} e^{-2\pi iht/q} \{1 + 2.3\,\Psi_o\,(2\pi ih/q) + 3.4\,\Psi_1(-3)\,4\pi^2 h^2/q^2\}$$

$$= F_7(t) + 2.3\,\Psi_o\,Cq\,(t) + 3.4\,\Psi_1(-3)\,C'q\,(t) - 4.5\,\Psi_2(-3)\,C''q(t)$$

with

[119] $$F_7(t) = \frac{q}{2\pi i} \sum_h \frac{e^{-2\pi iht/q}}{h}$$

so that

[120] $$|F_7(t)| < q/2\pi \sum^{q-1} 1/h = 0\,(q \log q) = 0\,(q^{1+\varepsilon})$$

Finally we obtain:

[121] $$\sum_{h=1}^{q-1} \Delta^{2)} \frac{e^{-At}}{A^3} = 0\,(q^{1+\varepsilon}) + 3C_q(t) + 0\,(q^{\varepsilon})$$

22. We now deal with the case of

$$\sum_h \Delta^{2)} \frac{te^{-At}}{A^2}$$

According to [105] we have:

$$\Delta^{2)} \; te^{-At} = t \; \Delta^{2)} \; e^{-At} \; \{ 1 + 0\,(1/t) \}$$

Hence

$$\sum_{h} \Delta^{2)} \frac{te^{-At}}{A^2} = t \sum \Delta^{2)} \frac{e^{-At}}{A^2} \; \{ 1 + 0\,(1/t) \}$$

We make now use of [114] in order to obtain:

[122] $$\sum_{\substack{h=1 \\ (h,q)=1}}^{q-1} \Delta^{2)} \frac{te^{-At}}{A^2} = t \; \{ C_q(t) + 0\,(q^{\varepsilon}) \} \; \{ 1 + 0\,(1/t) \}$$

The approximation

[123] $$\sum_{\substack{h=1 \\ (h,q)=1}}^{q-1} \Delta^{2)} \frac{te^{-At}}{A^2} = t \; \{ C_q(t) + 0\,(q^{\varepsilon}) \}$$

will suffice for our purposes.

23. We continue with the evaluation of the other sums along h, that appear in [87]. The following is:

[124] $$F_4(t) = \sum_{h=1}^{q-1} 1/A + \frac{e^{-At}}{A^2} \; (1 - e^{A})$$

In this case, it is sufficient with some raw approximation.
We have

[125] $$\sum_{h} |F_4(t)| < q \sum_{h=1}^{q-1} 1/2\pi h + q^2 \sum_{h=1}^{q-1} 2/4\pi^2 h^2 = 0\,(q \log q) + 0\,(q^2)$$

24. Next, we deal with the case of

$$[126] \quad F_8(t) + \sum_{q_1}^{[\sqrt{n}]} \sum_{h_1}^{q_1-1} \sum_{q_2}^{[\sqrt{n}]} \sum_{h_2}^{q_2-1} \frac{\mu(q_1)\ \mu(q_2)}{\varphi(q_1)\ \varphi(q_2)} \Delta^{2)} F_6(t)$$

with

$$F_6(t) = \frac{1}{A_2 - A_1} \left\{ \frac{\Delta^{2)} e^{-At}}{A_1} - \frac{\Delta^{2)} e^{-A_2 t}}{A_2} \right\}$$

The minimum value of $A_2 - A_1$ is $2\pi i\,(h_1/q_1 - h_2/q_2)$, where both fractions are contiguous Farey fractions.
It is known from ref. (3) that

$$\frac{h_1}{q_1} - \frac{h_2}{q_2} = 0 \left(\frac{1}{q^2} \right)$$

so that

$$\left| \frac{1}{A_2 - A_1} \right| < 0\,(q^2)$$

Besides,

$$\Delta^{2)}\ e^{-A_1 t} = 0\,(\,|\,e^{-A_1 t}\,|\,) = 0\,(1)$$

and hence

$$\left| \frac{\Delta^{2)}\ e^{-A_1 t}}{A_1} - \frac{\Delta^{2)}\ e^{-A_1 t}}{A_2} \right| < \left| \frac{1}{A_1} + \frac{1}{A_2} \right| = \left| \frac{1}{2\pi i h_1/q_1} + \frac{1}{2\pi i h_2/q_2} \right|$$

$$= \frac{1}{2\pi} \left\{ \frac{q_1}{h_1} + \frac{q_2}{h_2} \right\} < 0\,(\,q/h\,)$$

Consequently:

$$|F_6(t)| < 0(q^2)\ 0\left(\frac{q}{h}\right) = 0\left(\frac{q^3}{h}\right)$$

But:

$$\left|\sum_{h_2-1}^{q_2-1} \frac{\mu(q_2)}{\varphi(q_2)}\right| < \sum_{h_2-1}^{q_2-1} 0\left(\frac{q_2^{\ 3}}{h_2}\right) \frac{1}{\varphi(q_2)} < \frac{1}{\varphi(q_2)}\ 0(q_2^{\ 3} \log q_2)$$

and

$$\sum_{q_2=1}^{[\sqrt{n}]} \sum_{h_2=1}^{q_2-1} < \sum_{q_2=1}^{[\sqrt{n}]} 0(q_2^{\ 2} \log q_2 \ \log\ \log q_2) = 0(n^{3/2} \log n\ \log\ \log n)$$

The same thing is true for the sum extended upon the q_1 and h_1.
Hence

[127] $$|F_8(t)| < 0(n^3 \log^2 n (\log \log n)^2)$$

25. Having finished with the evaluation of the sums along h, we continue with the sums along q ($1 \leq q \leq [\sqrt{n}]$).
We begin by examining the term:

[128] $$T_1 = \Sigma \left(\frac{\mu(q)}{\varphi(q)}\right)^2 \Delta^{2)} F_2(t) = \sum_q \left(\frac{\mu(q)}{\varphi(q)}\right)^2 \sum_h \left\{ \Delta^{2)} \frac{te^{-At}}{A^2} + 2\ \Delta^{2)} \frac{e^{-At}}{A^3} \right\}$$

The value between brackets is, according to [121] and [123]:

[129] $$t\ C_q(t) + t\ 0(q^{\varepsilon}) + 3\ C_q(t) + 0(q^{1+\varepsilon})$$

or

[130] $$t\ C_q(t) + t\ R_q(t) + 3\ C_q(t) + 0(q^{1+\varepsilon})$$

with

[131] $$R_q(t) = 2.3\,\Psi_o \frac{2\pi i}{q} C'_q(t) + 3.4\,\Psi_1 (-3) \frac{(2\pi i)^2}{q^2} C''_q(t) + \ldots$$

Hence

[132] $$T_1 = t \sum_{q=1}^{[\sqrt{n}]} \left(\frac{\mu(q)}{\varphi(q)}\right)^2 C_q(t) + t \sum \left(\frac{\mu(q)}{\varphi(q)}\right)^2 0\,(q^{\varepsilon}) + 3 \sum \left(\frac{\mu(q)}{\varphi(q)}\right)^2 C_q(t)$$

$$+ \sum_{q=1}^{[\sqrt{n}]} \left(\frac{\mu(q)}{\varphi(q)}\right)^2 0\,(q^{1+\varepsilon})$$

$$= t \sum_{q=1}^{[\sqrt{n}]} \left(\frac{\mu(q)}{\varphi(q)}\right)^2 C_q(t) + t \sum \left(\frac{\mu(q)}{\varphi(q)}\right)^2 R_q(t) + 3 \sum_{q=1}^{[\sqrt{n}]} \left(\frac{\mu(q)}{\varphi(q)}\right)^2 C_q(t)$$

$$+ \sum_{q} \left(\frac{\mu(q)}{\varphi(q)}\right)^2 0\,(q^{1+\varepsilon})$$

But

$$\sum_{q=1}^{[\sqrt{n}]} \left(\frac{\mu(q)}{\varphi(q)}\right)^2 C_q(t)$$

is nothing but the singular series S_2 of Hardy-Littlewood truncated at $q = [\sqrt{n}]$. Hence we denote it by $S_2 [\sqrt{n}]$, so that

[133] $$S_2 (\sqrt{n}) = \sum_{q=1}^{[\sqrt{n}]} \left(\frac{\mu(q)}{\varphi(q)}\right)^2 C_q(t)$$

Then T_1 can be written as:

[134] $$T_1 = t\, S_2(\sqrt{n}) + t \sum_{q} \left(\frac{\mu(q)}{\varphi(q)}\right)^2 R_q(t) + 3\, S_2(\sqrt{n}) + \sum_{q} \left(\frac{\mu(q)}{\varphi(q)}\right)^2 0\,(q^{1+\varepsilon})$$

26. We now deal with the term

[135] $$T_2 = \sum \frac{\mu(q)}{\varphi(q)} \sum_h F_4(t)$$

$$T_2 = \sum_{q=1}^{[\sqrt{n}]} \frac{\mu(q)}{\varphi(q)} \{0(q^2)\} \qquad \text{(by [125])}$$

$$|T_2| <= \sum_{q=1}^{[\sqrt{n}]} 0 \left\{\frac{\log\log q}{q}\right\} 0(q^2) < \sum_{q=1}^{[\sqrt{n}]} 0(q \log\log q) = 0(n \log\log n)$$

27. The next case is

[136] $$T_3 = \sum_q \frac{1}{\varphi(q)} \sum_h \Delta^{2)} F_5(t) = \sum_{q=1}^{[\sqrt{n}]} \frac{1}{\varphi(q)} \{C_q(t) + 0(q^{\varepsilon})\}$$

by [115].

For $C_q(t)$, from [109], we obtain the majoration:

[137] $$|C_q(t)| < \sum_{\substack{h=1 \\ (h,q)=1}}^{q} 1 = \varphi(q)$$

so that

[138] $$|T_3| < \sum_{q=1}^{[\sqrt{n}]} 1 + \sum_{q=1} 0(q^{\varepsilon-1}) < 0(\sqrt{n})$$

28. The last thing we must evaluate in [87] is:

[139] $$F_{(n)} = \sum_{q=1}^{[\sqrt{n}]} \frac{\mu(q)}{\varphi(q)}$$

The function

[140] $$A_q = \frac{\mu(q)}{\varphi(q)}$$

is multiplicative. Hence, if $m = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, then holds:

[141] $$\sum_{d|m} \frac{\mu(d)}{\varphi(d)} = \left(1 - \frac{1}{\varphi(p_1)}\right)\left(1 - \frac{1}{\varphi(p_2)}\right) \dots \left(1 - \frac{1}{\varphi(p_k)}\right)$$

From [141] it is evident then that holds:

[142] $$\sum_{q=1}^{[\sqrt{n}]} \frac{\mu(q)}{\varphi(q)} = \prod_{p_i} \left(1 - \frac{1}{\varphi(p_i)}\right)$$

where the p_i denotes the different primes that are factors of the numbers that appear in the sequence 1, 2, 3, ... $[\sqrt{n}]$.
As $\varphi(p_i) = p_i - 1$, [138] turns out to be:

[143] $$\sum_{q=1}^{[\sqrt{n}]} \frac{\mu(q)}{\varphi(q)} = \prod_{p_i} \left(1 - \frac{1}{p_i - 1}\right)$$

The right-hand side product decreases as the quantity of factors increases. Hence the upper bound:

[144] $$\sum_{q=1}^{[\sqrt{n}]} \frac{\mu(q)}{\varphi(q)} < \delta_n < 1$$

where $\delta_n \rightarrow 0$ as $n \rightarrow \infty$.

29. We have now all the elements in order to evaluate $\nu(t)$ in formula [87], by using [127], [134], [138] and [144]. We get:

[145] $$\nu(t) = \delta_n^2 t + T_1 + 2A\ .\delta_n\, n^{9+1/4+\varepsilon} + 2.\delta_n T_2 + 2A\, n^{9+1/4+\varepsilon}\ T_3 + F_8(t)$$

$$= \delta_n^2 t + (t+3)\ S_2(\sqrt{n})\ t \sum \left(\frac{\mu(q)}{\varphi(q)}\right)^2 R_q(t) + 0(n^{\varepsilon})$$

$$+ 2A\,\delta_n\, n^{9+1/4+\varepsilon} + 2.\delta_n.0(n \log\log n) + 2A\, n^{9+1/4+\varepsilon+1/2} + 0(n^{3+\varepsilon})$$

$$= \delta_n^2 t + (t+3)\ S_2(\sqrt{n}) + t \sum_q \left(\frac{\mu(q)}{\varphi(q)}\right)^2 R_q(t) + 0(n^{3+\varepsilon})$$

30. We show now that the convergence of the singular series (using the majorations given in PN III), cannot be established (Hardy and Littlewood state that is convergent, but it is an evident mistake, whose origin is explained in the "Appendix" at the end of this paper).
We start with the facts (stated in PN III) that:

[146] $$C_q(t) = \begin{cases} p-1 & \text{if } p/t \\ -1 & \text{if } p/t \end{cases} \qquad C_q(t) < q$$

and $$\frac{1}{\varphi(q)} < e^{\gamma}\ \frac{\log\log q}{q}$$

Hence the series

$$\sum_{q=1}^{\infty} \frac{\mu^2(q)}{\varphi^2(q)}\ C_q(t) = \sum_{q=1}^{\infty} A_q$$

is majorized by

$$e^{2\gamma} \sum_{q=1}^{\infty} \frac{q}{q^2}\ (\log\log q)^2$$

whose divergence is obvious.

Hence, in order to evaluate the singular series when extended to $q = \infty$, we must perform a by-pass.

For any multiplicative function A_n we have, in general, that:

$$\sum_{n=1}^{\infty} \frac{A_n}{n^s} = \prod_p \left\{ 1 + \frac{A_p}{p^s} + \frac{A_{p2}}{p^{2s}} + \dots \right\}$$

but, in our case, due to the presence of $\mu(n)$ in A_n , this reduces to:

$$\sum_{n=1}^{\infty} \frac{A_n}{n^s} = \prod_p \left\{ 1 + \frac{A_p}{p^s} \right\} \qquad \text{(p: primes)}$$

For $s = o$ the left hand side series presumably diverges, but the right hand side product converges, because:

[147] $$\prod_p (1 + A_p) = \prod_{p \backslash t} (1 + A_p) \prod_{p \not\backslash t} (1 + A_p)$$

$$= \prod_{p \backslash t} \left\{ 1 + \frac{1}{p-1} \right\} \prod_{p \not\backslash t} \left\{ 1 - \frac{1}{(p-1)^2} \right\} = \prod_{p \backslash t} \frac{1 + \frac{1}{p-1}}{1 - \frac{1}{(p-1)^2}} \prod_p \left\{ 1 - \frac{1}{(p-1)^2} \right\} =$$

$$= 2 \prod_{p=3}^{\infty} \left\{ 1 - \frac{1}{(p-1)^2} \right\} \prod_{\substack{p \backslash t \\ p \neq 2}} \frac{p-1}{p-2}$$

is plainly convergent.

31. In formula [145] we can now choose n as low as $[\log_r t]$, where $\log_r t$ means the r - th iterated logarithm of t, in order to obtain the unconditional result.

[148] $$v(t) = (t+3)\ S_2([\sqrt{\log_r t}]) + t \sum_{q=1}^{\sqrt{n}} \left(\frac{\mu(q)}{\varphi(q)} \right)^2 R_q(t) + 0(\log_r^{3+\varepsilon} t)$$

with $R_q(t)$ defined by [131].

At present, we can prove only that the second term is slightly higher than the first. If we could improve the majoration [116] in order to transform the exponent $+\varepsilon$ in $-\varepsilon$, then we could prove easily conjecture A, that

[149] $$v(t) \sim t\ S_2(\infty)$$

where, in the case that $S_2(\infty)$ be divergent, we could replace it by the infinite product [147], that is its analytic continuation (in the original work of H-L, the error was $n^{1/4}$ in change of n^{ε}).

32. However, we dispose still of other alternative.

On the ground of what has been seen before, we can write:

[150] $$v(t) \sim \sum_{q=1}^{[\sqrt{n}]} \frac{\mu^2(q)}{\varphi^2(q)} \sum_h \Delta^{2)} \frac{t}{A^2} e^{-At}$$

By [105], we have equally well that

[151] $$v(t) \sim t \sum_q \frac{\mu^2(q)}{\varphi^2(q)} \sum_h \Delta^{2)} \frac{e^{-At}}{A^2}$$

In [89] we have stated a result derived from Carr's book, i.e. that

[89] $$\Delta^{2)} e^{-At} = - \left\{ 2 \sin \frac{|A|}{2} \right\}^2 e^{-A(t+1)}$$

This formula, however, requires a slight modification in our case, because Carr defines $\Delta^{1)}$ as:

$$\Delta^{1)}\ f(x) = f(x+1) - f(x)$$

while we have in change:

$$\Delta^{1)}\ f(x) = f(x) - f(x-1)$$

Hence, we must replace [89] by:

[152] $$\Delta^{2)}\ e^{-At} = -\left\{ 2 \sin \frac{|A|}{2} \right\}^2 e^{-At}$$

Inserting this in [151] follows:

[153] $$v(t) \sim t \sum_q \frac{\mu^2(q)}{\varphi^2(q)} \sum_h \left\{ \frac{q}{\pi h} \sin \frac{\pi h}{q} \right\}^2 e^{-2\pi iht/q}$$

Now, the function of h and q

$$\varepsilon_{q,h} = \frac{q}{\pi h} \sin \frac{\pi h}{q}$$

decreases steadily in the interval $0 < h < q$, and we have

$$0 < \varepsilon_{q,h} \leq 1$$

in the interval.
Hence we can state that holds <u>unconditionally</u>

[154] $$v(t) \sim t \sum_{q=1}^{[\sqrt{n}]} \frac{\mu^2(q)}{\varphi^2(q)} \sum_{h=o}^{q-1} \varepsilon_{q,h}\ e^{-2\pi iht/q}$$

This looks very similar to conjecture A, that

[155] $$v(t) \sim t \sum_{q=1}^{\infty} \frac{\mu^2(q)}{\varphi^2(q)} \sum_{h=o}^{q-1} e^{-2\pi iht/q}$$

only that as the function $\varepsilon_{q,h}$ is not multiplicative, the series in [154] cannot be transformed into a product.
On the other hand, perhaps, the series in [154] have more chance to be convergent as $n \to \infty$ than the series in [155], because their terms in the sum along h have smaller modulus than the terms in the inner sum of [155].

33. After all the long calculation performed above, we conclude that the circle method is a particular way of finding the inverse transform in the fundamental formula [61].
In fact, given the function

$$f(x) = \sum \log p .\ x^p$$

the circle method consists in approximating $f(x)$ in the vicinity of their singularities on the unitary circle by means of a Farey dissection of order $[\sqrt{n}]$; to square the resulting expression, and to integrate along a continuous contour, formed by the Farey arcs.
When performing the substitution $x = e^{-s}$ in our method of the Laplace transform, we use the function

$$\mathcal{L}\ \{ \vartheta(x)\} = \frac{1}{s} \sum_{p} \log p .\ e^{-ps}$$

that has an infinitude of singularities on the line $s = \sigma = o$. Once we have written the pertinent Farey dissection, we square it. Now, if we should follow the circle method, we should integrate along a Bromwich contour formed by the Farey arcs, changed by the change of variable $x = e^{-s}$.
This is nothing but the complex inversion formula for $\mathcal{L}^{-1}$, when applied to that special contour.
But the former calculations indicate that it is not mandatory to follow such procedure: the value of $\mathcal{L}^{-1}$ can be obtained more easily through the use of tables of elementary transforms.
As regards the calculus of the second differences, we have seen that do not appear any serious difficulty.
We deduce then that the method of the Laplace transform is, at least in this case, much more general and powerful than the circle method, and that, furthermore, allows us to make a complete

discussion of the remainder terms. On the other hand, we do not need to make any hypothesis concerning the position of the zeros of the L - series or of the zeta function.

34. We examine now the question if for the effective evaluation of the function ν (t) is indispensable the knowledge of the infinite singularities of the function Σ lop p. e^{-ps}, the corresponding Farey dissection, etc.
The answer is absolutely negative. As Hardy and Littlewood point out in PN III, the function ν (t) has the same asymptotic value than the function S (t) of paragraph 1 of this paper. But this last can be expressed in exact and asymptotic form through the zeros of the zeta function, according to formula [28], without the use of singular series.
It is then entirely false the ill-disposed comment of Prof. Vaughan in "Mathematical Reviews" according to which the majority of the formulas in my paper ref. (8) are false because it has been ignored the existence of the singularities on $\sigma = o$.

* * * * * * * * * * *

APPENDIX

In p. 26 § 3.16 of PN III, Hardy and Littlewood state: "In order to complete the proof of Theorem A, we have merely to show that the finite series in (3.156) may be replaced by the infinite series S_1.
Now

$$[156] \qquad \left| n^{r-1} \sum_{q>N} \left(\frac{\mu(q)}{\varphi(q)} \right)^r C_q(-n) \right| \quad B\, n^{r-1} \sum_{q>N} q^{1-r} (\log q)^B < B\, n^{r/2} (\log n)^B$$

and

$$r/2 < r-1 + 9-3/4 \text{''}$$

Let us decompose this calculation, step by step. Here they use the following bounds:

$$\text{I)} \qquad \frac{1}{\varphi(q)} < B \frac{(\log q)^B}{q} \qquad\qquad \text{(Lemma 10, p.21)}$$

(though they could have been used the better bound that appears two lines below:

$$\frac{1}{\varphi(q)} < e^{\gamma} \frac{\log \log q}{q} \quad)$$

II) $C_q(-n) < q-1 < q$

III) $| \mu^r(q) | \leq 1$

On these grounds, we have the following majoration for $r = 2$

$$\left| \sum_{q>N} \left(\frac{\mu(q)}{\varphi(q)} \right)^2 C_q(-n) \right| < \sum_{q>N} B^2 \frac{(\log q)^{2B}}{q^2} q < B^2 \sum_{q>N} \frac{(\log n)^{2B}}{q}$$

whose divergence is obvious.
Hence the finite series cannot be replaced by the infinite series, and Lemma 12 in p.27 is proved only for $r \geq 3$.
This has as a consequence that their transformation of the singular series into a product in p.28, formula (3.226) has only a formal character and is not a proof.
The mistake of H-L in [156] was to put

$$B\, n^{r-1} \sum_{q>r} q^{1-r} (\log q)^B < B\, n^{r/2} (\log n)^B$$

The term n^{r-1} cannot be introduced into the sum, as it represents the order of magnitude of $\nu(n)$. Hence, the correct calculation is:

$$B\, n^{r-1} \sum_{q>N} q^{1-r} (\log q)^B < B_1\, n^{r-1} \frac{N^{2-r}}{2-r} (\log N)^B \qquad \underline{\text{if } r \neq 2}$$

and, as $N < \sqrt{n}$, this is

$$< B_2\, n^{r-1}\, n^{1-r/2} (\log n)^B = B_2\, n^{r/2} (\log n)^B \qquad \underline{\text{except when } r=2}$$

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