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THE GOLDBACH PROBLEM (II)

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ABSTRACT

It is studied in first place the function $s(t) = \sum\sum \log p_1 \log p_2$, where the sum is extended for all prime numbers p_i such that $p_1^m + p_2^n = t$. Are proed formulas (14) and (26), which express its value in terms of Chebyshev's function $\psi(x)$.

In this way is obtained formula (38), that gives the asymptotic value of $s(t)$ with a new "singular" series which runs through the zeros of the Zeta function, but that at present can not be evaluated in a sufficiently accurate form.

In second place, for the function $\nu(t) = \sum\sum \log p_1 \log p_2$ (with $p_1 + p_2 = t$), already considered by Hardy-Littlewood in "Partitio Numerorum III" (P.N.III), is proved the exact formula

$$\frac{\nu(t+0) + \nu(t-0)}{2} = \Delta^{2)} \mathcal{L}^{-1}\{\mathcal{L}^2\{\vartheta(x)\}\}$$

- where $\Delta^{2)}$ = second difference;
- $\mathcal{L}$ and $\mathcal{L}^{-1}$ are the direct and inverse Laplace transforms;
- $\vartheta(x) = \sum \log p \quad (p \leq x)$.

The circle method applied in P.N.III is equivalent to determine $\mathcal{L}^{-1}$ through the comlpex inversion formula along a Bromwich contour. But it is evident that is much preferable to employ tables of direct and inverse transforms because the functions involved are elementary; because is obtained an exact expression for the remainder, and because all the calculus is by far more simple.

One arrive thus to the inconditional formula (130), which very closely resembles the famous conjecture A of P.N.III.

§1. As is well known, Goldbach's problem begins in 1742, in a letter where Chr. Goldbach points out to Euler that apparently all even numbers can be represented as the sum of two primes.

In this paper, instead of determining the quantity of decompositions $G(t) = \sum\sum 1$, where $p_1 + p_2 = t$, we calculate

$$s(t) = \sum\sum \log p_1 \log p_2 \qquad (m \geq 1, \quad n \geq 1)$$
$$p_1^m + p_2^n = t$$

$$s(t) = \sum_{p_1^m + p_2^n = t}\sum \log p_1 \log p_2 \qquad (m \geq 1, \quad n \geq 1)$$

and afterwards the function $\nu(t)$:

$$\nu(t) = \sum\sum_{p_1+p_2=t} \log p_1 \log p_2$$

According to P.N.III, we have

$$s(t) \sim \nu(t) \qquad G(t) \sim \frac{\nu(t)}{\log^2 t}$$

We begin with Chebyshev's function:

$$\psi(x) = \sum_{p^m \leq x} \log p = \sum_{n \leq x} \lambda(n) \tag{1}$$

where p denotes the prime numbers, and $\lambda(n)$ is von Mangoldt's function:

$$\begin{cases} \lambda(n) = \log p & \text{if } n = p^m \qquad (m = 1, 2, 3, \ldots) \\ \lambda(n) = 0 & \text{in every other case} \end{cases} \tag{2}$$

For the function

$$\frac{\psi(x+0)+\psi(x-0)}{2} = \begin{cases} \sum_{n\leq x} \lambda(n) & \text{if } x \text{ is an integer} \\ \sum_{n \leq x} \lambda(n) - \frac{1}{2}\lambda(x) & \text{if } x \text{ is not an integer} \end{cases} \tag{3}$$

we have the following development [1]:

$$\frac{\psi(x+0)+\psi(x-0)}{2} = x - \sum_{\rho} \frac{x^\rho}{\rho} - \frac{1}{2}\log(1-\frac{1}{x^2}) - \frac{\zeta'(0)}{\zeta(0)} \tag{4}$$

where the ρ's denote the infinite imaginary zeros of the Zeta function, being $\rho_\nu = \beta_\nu + i\gamma_\nu$

It is known also that (2):

$$\frac{\psi(x+0)+\psi(x-0)}{2} = x - \sum_{|\gamma|<T} \frac{x^\rho}{\rho} + o(\frac{x}{T}\sqrt{\log x}) \tag{5}$$

Taking into account formula (3), we form the new function $\varphi(x)$:

$$\varphi(x) = \frac{\psi(x+0)+\psi(x-0)}{2} - \frac{\psi(x-1+0)+\psi(x-1-0)}{2} \tag{6}$$

and find that

$$\varphi(x) = \begin{cases} \lambda(x) & \text{if } p^r < x < p^r + 1 \\ \frac{1}{2}\lambda(x) & \text{if } x = p^r \\ 0 & \text{in any other case} \end{cases} \tag{7}$$

with an analogous folmula for $\varphi(t-x)$.

Next, we define other function $I(x) = \varphi(x)\varphi(t-x)$ that according to (7) assumes the following values:

$$I(x) = \begin{cases} i) \ \lambda(x)\lambda(t-x) & \text{if } p_1^m < x < p_1^m + 1 \\ & \text{and besides} \\ & p_2^n < t - x < p_2^n + 1 \\ ii) \ \frac{1}{4}\lambda(x)\lambda(t-x) & \text{if } x = p_1^m, \quad t - x = p_2^n \\ iii) \ 0 & \text{in any other case} \end{cases} \tag{8}$$

If now denote with $s(t)$ to the function

$$s(t) = \sum_{x=1}^{t-1} \lambda(x)\lambda(t-x) = \sum_{p_1^m + p_2^n = t}\sum \log p_1 \log p_2 \tag{9}$$

we arrive to the function $g_2(t)$ already considered by Hardy and Littlewood in P.N.III, ref. [3], p.38 in connection with this problem.

Due to (8), we can evaluate $s(t)$ in two different ways:

- 1) In a discrete way, according to (8) ii) is

$$s(t) = \frac{1}{4}\sum_{x=1}^{t-1} I(x) \tag{10}$$

- 2) In a continuous way, using (8) i):

$$s(t) = \int_1^{t-1} I(x)dx \tag{11}$$

The discret variant finds overwhelming difficulties in the further calculation; in change the continuous variant is perfectly accessible and opens a new way essentially different from that used Hardy and Littlewood.

In fact, we have that

$$s(t) = \int_1^{t-1} I(x)dx = \int_1^{t-1} \varphi(x)\varphi(t-x)dx \tag{12}$$

This can be written also as:

$$s(t) = \int_0^t \varphi(x)\varphi(t-x)dx \tag{13}$$

because in the intervals $0 < x < 1$ and $t-1 < x < t$ the integrand kanishes.

It is evident at first sight that in (13) we have an integral of convolutory type.

Here again, we have two ways in order to evaluate $s(t)$:

• 1) To replace in (13) the values of φ and ψ given by the formulas (6) and (5), and perform after the product $\varphi(x)\varphi(t-x)$ and the resultant integration.

• 2) To take the Laplace transform of $s(t)(=\mathcal{L}\{s(t)\})$, and make use of the known property of the convolutory integrals:

$$\mathcal{L}\{s(t)\} = \mathcal{L}^2\{\varphi(x)\} \tag{14}$$

The right hand member is evaluated after by means of (6) and (5), and finally $s(t)$ is determined through the inverse transform.

§2. We give here a brief sketch of the first variant. One has:

$$s(t) = \int_1^{t-1}\{1 - \sum_{|\gamma|<T}\frac{x^\rho-(x-1)^\rho}{\rho} + o(\frac{x}{T}\sqrt{\log x}) + o(\frac{1}{x^2})\} \tag{15}$$

$$\{1 - \sum_{|\gamma|<T}\frac{(t-x)^\rho-(t-x-1)^\rho}{\rho} + o(\frac{t-x}{T}\sqrt{\log(t-x)}) + o(\frac{1}{(t-x)^2})\}dx$$

After some manipulation, this reduces to:

$$s(t) = t-2-2\int_1^{t-1}D_1dx + \int_1^{t-1}D_1D_2dx + o(1) + o(\frac{1}{T})\int_1^{t-1}x\sqrt{\log x}dx-$$

$$-\int_1^{t-1}D_1 o(\frac{1}{(t-x)^2})dx + \int_1^{t-1}D_2 o(\frac{x}{T}\sqrt{\log x})dx+$$

$$+o(\frac{1}{T^2})\int_1^{t-1}o\{x(t-x)\sqrt{\log x.\log(t-x)}\}dx + \int_1^{t-1}o\{\frac{x\sqrt{\log x}}{T(t-x)^2}\}dx$$

where

$$D_1 = \sum_{|\gamma|<T}\frac{x^\rho-(x-1)^\rho}{\rho} \qquad D_2 = \sum_{|\gamma|<T}\frac{(t-x)^\rho-(t-x-1)^\rho}{\rho}$$

But

$$\int_1^{t-1}D_1dx = \sum_{|\gamma|<T}\frac{t^{\rho+1}-(t-1)^\rho}{\rho(\rho+1)} = o(t^\vartheta\log^2 t) \quad \text{for every T} \tag{16}$$

$$\int_1^{t-1}D_1D_2dx = \sum_{|\gamma_1|<T}\sum_{|\gamma_2|<T}\frac{B(\rho_1+1,\rho_2+1)}{\rho_1\rho_2}\Delta^{2)}t^{\rho_1+\rho_2+1} + r(t,T) \tag{17}$$

where $B(x,y)$ is the Beta function of Euler;

$$\begin{aligned}\Delta^{2)}t^{\rho_1+\rho_2+1} &= t^{\rho_1+\rho_2+1} - 2(t-1)^{\rho_1+\rho_2+1} + (t-2)^{\rho_1+\rho_2+1} \\ &= \text{ second difference of } t^{\rho_1+\rho_2+1}\end{aligned}$$

and

$$r(t,T) = 2\sum_{|\gamma_1|<T}\sum_{|\gamma_2|<T}\frac{t^{\rho_1+\rho_2+1}}{\rho_1\rho_2}\int_0^{1/t} y^{\rho_1}\{(1-y-\frac{1}{t})^{\rho_2} - (1-y)^{\rho_2}\}dy$$

The modulus of the integral is smaller than

$$2\int_0^{1/t} y^{\vartheta}(1-y)^{\vartheta}dy < 2\int_0^{1/t} y^{\vartheta}dy = \frac{2}{(\vartheta+1)t^{\vartheta+1}}$$

Hence:

$$r(t,T) < \frac{1}{t^{\vartheta}}o\{\sum_{\gamma<T}\sum_{\gamma<T}\frac{t^{\rho_1+\rho_2}}{\rho_1\rho_2}o(1)\} = \frac{1}{t^{\vartheta}}o(t^{\vartheta}\log^2 t)^2 = o(t^{\vartheta}\log^4 t) \tag{18}$$

Further, we have

$$|\int_1^{t-1} D_1 o(\frac{1}{(t-x)^2})dx| < c_0\int_1^{t-1}\frac{x^{\vartheta+\epsilon}}{(t-x)^2}dx < \tag{19}$$

$$< t^{\vartheta+\epsilon-1}\int_0^1\frac{y^{\vartheta+\epsilon}}{(1-y)^2}dy = o(t^{\vartheta+\epsilon-1})$$

The other terms in (15) tend to zero as $T\to\infty$; hence we deduce:

$$s(t) = t + o(t^{\vartheta}\log^4 t) + \sum_{\rho_1}\sum_{\rho_2}\frac{B(\rho_1+1,\rho_2+1)}{\rho_1\rho_2}\Delta^{2)}t^{\rho_1+\rho_2+1} \tag{20}$$

if the double series is convergent. This question is to be discussed in paragraph 6 later. (In the preceding lines $\vartheta = \sup\beta_\nu$; $\rho_\nu = \beta_\nu + i\gamma_\nu$.)

§3. The second method, using formula (14), is nothing but a scholar exercise involving the Laplace transforms.

In order to simplifay operations, we begin by writting (5) as:

$$f(x) = \frac{\psi(x+0)+\psi(x-0)}{2} = x - \sum_{|\gamma|<T}\frac{x^\rho}{\rho} + o(\frac{x^{\epsilon+1}}{T}) \tag{21}$$

with ε arbitrarily small.

Then we have:

$$\mathcal{L}\{\varphi(x)\} = \mathcal{L}\{f(x)\} = \mathcal{L}\{f(x-1)\} \tag{22}$$

But

$$\mathcal{L}\{f(x-1)\} = \int_0^\infty e^{-sx} f(x-1)dx = \int_{-1}^\infty e^{-s(y+1)} f(y)dy = \tag{23}$$

$$= e^{-s}\int_0^\infty e^{-sy} f(y)dy = e^{-s}\mathcal{L}\{f(x)\}$$

From here, replacing in (22) we obtain:

$$\mathcal{L}\{\varphi(x)\} = (1-e^{-s})\mathcal{L}\{f(x)\} \tag{24}$$

Now, $f(x)$ and $\psi(x)$ differ between themselves by a null function; hence their Laplace transforms are the same, and we can put:

$$\mathcal{L}\{\varphi(x)\} = (1-e^{-s})\mathcal{L}\{\psi(x)\} \tag{25}$$

Replacing this value in (14) we derive the fundamental equation:

$$\mathcal{L}\{s(t)\} = (1-e^{-s})^2\mathcal{L}^2\{\psi(x)\} \tag{26}$$

Due to the fact that $s(t)$ is a discontinuous functions with jumps of the first class, we deduce:

$$\frac{s(t+0)+s(t-0)}{2} = \mathcal{L}^{-1}\{(1-e^{-s})^2\mathcal{L}^2\{\psi(x)\}\} \tag{27}$$

But a rule of manipulation of the inverse transform $\mathcal{L}^{-1}$ says that if

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

then holds that

$$\Delta^{m)} f(t) = \mathcal{L}^{-1}\{(1-e^{-s})^m F(s)\}$$

and reciprocally, where $\Delta^{m)}$ denotes the m-th difference of $F(t)$.

Applying this property to (27) we obtain finally

$$\frac{s(t+0)+s(t-0)}{2} = \Delta^{2)}\mathcal{L}^{-1}\{\mathcal{L}^2\{\psi(x)\}\} \tag{28}$$

Now it is evident this the Fundamental problem is the calculation of $\mathcal{L}\{\psi(x)\}$.

§4. Next, we have two ways in order to evaluate $\mathcal{L}\{\psi(x)\}$:

• a) The most obvious one is to determine this transform through the use of formula (5) or (21);

• b) The other alternative is to determine the branch points of $\mathcal{L}\{\psi(x)\}$ and afterwards to deduce the value of $\mathcal{L}^{-1}$ by the current methods of complex variable or more simple still, by the use of tables.

§5. We develop in first place the method described in a) above. We have, by (21) and the property of lineality of the transform, when applied to a finite sum:

$$\mathcal{L}\{\psi(x)\} = \mathcal{L}\{x\} - \mathcal{L}\{\sum_{|\gamma|<T} \frac{x^\rho}{\rho}\} + \frac{1}{T}o(\{\mathcal{L}(x^{\varepsilon+1})\} = \tag{29}$$

$$= \mathcal{L}\{x\} - \sum_{|\gamma|<T} \frac{1}{\rho}\mathcal{L}\{x^\rho\} + \frac{1}{T}o\{\mathcal{L}(x^{\varepsilon+1})\}$$

As is well known:

$$\mathcal{L}\{t^\alpha\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}} \tag{30}$$

so that from (29) follows:

$$\mathcal{L}\{\psi(x)\} = \frac{1}{s^2} - \sum_{|\gamma|<T} \frac{\Gamma(\rho+1)}{\rho s^{\rho+1}} + \frac{1}{T}o(\frac{1}{s^{2+\varepsilon}}) \tag{31}$$

Squaring both sides:

$$\mathcal{L}^2\{\psi(x)\} = \frac{1}{s^4} - \sum\sum \frac{\Gamma(\rho_1)\Gamma(\rho_2)}{s^{\rho_1+\rho_2+2}} + \frac{1}{T^2}o(\frac{1}{s^{4+2\varepsilon}}) - \tag{32}$$

$$-\frac{2}{s^2}\sum \frac{\Gamma(\rho)}{s^{\rho+1}} + \frac{1}{T}o(\frac{1}{s^{4+\varepsilon}}) - \frac{1}{T}o(\frac{1}{s^{2+\varepsilon}})\sum \frac{\Gamma(\rho)}{s^{\rho+1}}$$

Taking the inverse transforms of the right hand side (and here again is of application (30)), we deduce:

$$\mathcal{L}^{-1}\{\mathcal{L}^2\{\psi(x)\}\} = \tag{33}$$

$$\frac{t^3}{3!} - \sum_{|\gamma_1|<T}\sum_{|\gamma_2|<T} \frac{\Gamma(\rho_1)\Gamma(\rho_2)}{\Gamma(\rho_1+\rho_2+2)} t^{\rho_1+\rho_2+1} + \frac{1}{T^2}o\{t^{3+2\varepsilon}\} -$$

$$-2\sum_{|\gamma|<T} \frac{\Gamma(\rho)}{\Gamma(\rho+3)} t^{2+\rho} + \frac{1}{T}o\{t^{\varepsilon+3}\} - \frac{1}{T}\sum_{|\gamma|<T} \frac{\Gamma(\rho)}{\Gamma(\rho+\varepsilon+3)} o(t^{\rho+\varepsilon+2})$$

Hence by (28)

$$\frac{s(t+0)+s(t-0)}{2} = \tag{34}$$

$$\Delta^{2)}\frac{t^3}{3!} - \sum_{|\gamma_1|<T}\sum_{|\gamma_2|<T} \frac{B(\rho_1+1,\rho_2+1)}{\rho_1\rho_2} \Delta^{2)} t^{\rho_1+\rho_2+1} +$$

$$+\frac{1}{T^2}\Delta^{2)}o\{t^{2\varepsilon+3}\} - 2\sum_{|\gamma|<T} \frac{\Delta^{2)}t^{2+\rho}}{(\rho+2)(\rho+1)\rho} + \frac{1}{T}\Delta^{2)}o\{t^{\varepsilon+3}\} -$$

$$-\frac{1}{T}\sum_{|\gamma|<T}\frac{\Gamma(\rho)}{\Gamma(\rho+\varepsilon+3)}\,\Delta^{2)}\,o(t^{\rho+\varepsilon+2})$$

where again $B(x,y)$ is Euler's Beta function and

$$\Delta^{2)}\frac{t^3}{3!}=\frac{1}{3!}(t^3-2(t-1)^3+(t-2)^3)=t-1 \tag{35}$$

As regards $\Delta^{2)}t^{2+\rho}$ we have:

$$\Delta^{2)}t^{2+\rho}=t^{2+\rho}-2(t-1)^{2+\rho}+(t-2)^{2+\rho}= \tag{36}$$

$$=t^{2+\beta}.t^{i\gamma}-2(t-1)^{2+\beta}(t-1)^{i\gamma}+(t-2)^{2+\beta}(t-2)^{i\gamma}=$$

$$=t^{2+\beta}\{t^{i\gamma}-2(1-\frac{1}{t})^{2+\beta}(t-1)^{i\gamma}+(1-\frac{2}{t})^{2+\beta}(t-2)^{i\gamma}\}=$$

$$=t^{2+\beta}\,\Delta^{2)}\,t^{i\gamma}\{1+o(\frac{1}{t})\}=-\gamma^2t^{\beta}t^{i\gamma}\{c_0-o(\frac{1}{t})\}$$

The last step will be proved in paragraph 14 later. It follows that

$$|\sum_{|\gamma|<T}\frac{\Delta^{2)}t^{2+\rho}}{(\rho+2)(\rho+1)\rho}|<\;c_1\,|\sum_{|\gamma|<T}\frac{t^{\rho}}{\rho}|=\;o(t^{\vartheta}\log^2 t) \tag{37}$$

Consequently, letting $T\to\infty$ in (34) we get:

$$\frac{s(t+0)+s(t-0)}{2}= \tag{38}$$

$$t-1+o(t^{\vartheta}\log^2 t)+\sum_{\rho_1}\sum_{\rho_2}\frac{B(\rho_1+1,\rho_2+1)}{\rho_1\rho_2}\,\Delta^{2)}\,t^{\rho_1+\rho_2+1}$$

which looks very similar to (20), but that however does not keep complete coincidence.

The reason is the following: we have treated the variable t as if it where continuous. But this is not the case, as t is always an integer number. Hence, the inverse transform of (30), which currently is writen as

$$\mathcal{L}^{-1}\{\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}\}=t^{\alpha} \tag{39}$$

must be written in our case

$$\mathcal{L}^{-1}\{\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}\}=\frac{(t+0)^{\alpha}+(t-0)^{\alpha}}{2} \tag{40}$$

When this value is placed at the right hand side of (38) in place of t^{α} everywhere, we can then equate the terms with $(t+0)$ and $(t-0)$ in both members in order to obtain:

$$s(t) = t + o(t^{\vartheta} \log^2 t) + \sum_{\rho_1} \sum_{\rho_2} \frac{B(\rho_1 + 1, \rho_2 + 1)}{\rho_1 \rho_2} \Delta^{2)} t^{\rho_1 + \rho_2 + 1} \tag{41}$$

that is exactly (20), apart from an irrelevant exponent in the log term.

§6. Convergence of the double series

The asymptotic formula for $| \Gamma(\sigma + it) |$ (for large values of t) is

$$| \Gamma(\sigma + it) | \sim \sqrt{2\pi} \, | t |^{\sigma - 1/2} e^{-\frac{\pi | t |}{2}} \tag{42}$$

Hence we have:

$$| \frac{B(\rho_1 + 1, \rho_2 + 1)}{\rho_1 \rho_2} | = | \frac{\Gamma(\rho_1)\Gamma(\rho_2)}{\Gamma(\rho_1 + \rho_2 + 2)} | \sim \tag{43}$$

$$\sim \sqrt{2\pi} e^{-\frac{\pi | \gamma_1 |}{2} - \frac{\pi | \gamma_2 |}{2} + \frac{\pi | \gamma_1 + \gamma_2 |}{2}} \frac{| \gamma_1 |^{\beta_1 - 1/2} | \gamma_2 |^{\beta_2 - 1/2}}{| \gamma_1 + \gamma_2 |^{\beta_1 + \beta_2 + 3/2}}$$

which is valid unless $\gamma_1 = -\gamma_2$.

Now, actually, this case occurs, as the zeros of the Zeta function appear in conjugate pairs $\rho, \overline{\rho}$. Let us calculate the influence of these pairs:

$$\sum_{\rho} \sum_{\overline{\rho}} \frac{| \Gamma(\rho) |^2}{| \Gamma(\beta_1 + \beta_2 + 2) |} = o(\sum_{\gamma_i} e^{-\pi \gamma_i}) = o(1)$$

hence its influence is negligible.

The exponential term in (43) always appears, unless that

$$| \gamma_1 | + | \gamma_2 | = | \gamma_1 + \gamma_2 | \tag{44}$$

As obviously we have the bound

$$\Delta^{2)} t^{\rho_1 + \rho_2 + 1} = o(t^{2\vartheta + 1})$$

for every ρ_1 and ρ_2, it is evident that the sum

$$\sum \sum \frac{\Gamma(\rho_1)\Gamma(\rho_2)}{\Gamma(\rho_1 + \rho_2 + 2)} \Delta^{2)} t^{\rho_1 + \rho_2 + 1}$$

when extended to those values for which $| \gamma_1 | + | \gamma_2 | \neq | \gamma_1 + \gamma_2 |$ is formed by exponential terms that are quickly convergent and presumibly small.

Hence the dominant terms are those for which (44) holds. In order that (44) hold, we must have $\gamma_1 > 0$, $\gamma_2 > 0$ or $\gamma_1 < 0$, $\gamma_1 < 0$.

Hence the terms with negative γ's duplicate the terms with positive γ's, and it is sufficient to analyze this last case.

Under this assumption, from (43) flows:

$$\sum_{|\gamma_1|<T}\sum_{|\gamma_2|<T} \left| \frac{\Gamma(\rho_1)\Gamma(\rho_2)}{\Gamma(\rho_1+\rho_2+2)} \right| \sim \sum_{|\gamma_1|<T}\sum_{|\gamma_2|<T} \frac{\gamma_1^{\beta_1-1/2}\gamma_2^{\beta_2-1/2}}{(\gamma_1+\gamma_2)^{\beta_1+\beta_2+3/2}} \tag{45}$$

Taking into account that $\gamma = o(n^{1-\epsilon})$ and using Cauchy's criterion for the majoration of series by means of integrals, we deduce that this series has the same order of magnitude than

$$\int_{c_1}^{T}\int_{c_2}^{T} \frac{u^{(\beta_1-1/2)(1-\epsilon)}v^{(\beta_2-1/2)(1-\epsilon)}}{(u^{1-\epsilon}+v^{1-\epsilon})^{\beta_1+\beta_2+3/2}} du dv$$

Performing the substitutions $u^{1-\epsilon} = x$; and $v^{1-\epsilon} = y$ this transforms to:

$$\int_{c_3}^{T^{\epsilon+1}}\int_{c_4}^{T^{\epsilon+1}} \frac{x^{(\beta_1-1/2)(-\frac{\varepsilon}{1-\varepsilon})}y^{(\beta_2-1/2)(-\frac{\varepsilon}{1-\varepsilon})}(1-\varepsilon)^2}{(x+y)^{\beta_1+\beta_2+3/2}} dx dy$$

This is smaller than

$$\int_{c_3}^{T^{\epsilon+1}}\int_{c_4}^{T^{\epsilon+1}} \frac{x^{(\beta_1-1/2)}y^{(\beta_2-1/2)}}{(x+y)^{\beta_1+\beta_2+3/2}} dx dy \tag{46}$$

Now

$$\int_0^T \frac{x^{\alpha_1}}{(x+y)^{\delta}} dx \sim \frac{T^{\alpha_1}}{(T+y)^{\delta-1}(\alpha+1-\delta)} \quad (\text{ ref. [4] })$$

and on the same grounds

$$\int_0^T \frac{y^{\alpha_2}T^{\alpha_1}}{(T+y)^{\delta-1}(\alpha+1-\delta)} dy \sim \frac{T^{\alpha_1}T^{\alpha_2}}{(2T)^{\delta-1}(\alpha_2-\delta)(\alpha_1+1-\delta)}$$

so that (46) is asymptotically equal to

$$\frac{T^{(\alpha_1+\alpha_2)(\epsilon+1)-\delta+1}}{2^{\delta-1}(\alpha_2-\delta)(\alpha_1+1-\delta)}$$

As $\alpha_1 = \beta_1 - 1/2$ and $\alpha_2 = \beta_2 - 1/2$ and $\delta = \beta_1 + \beta_2 + 3/2$ the preceding expression has the value

$$o(T^{(\epsilon+1)(\beta_1+\beta_2-1)-(\beta_1+\beta_2+3/2)}) = o(T^{-1/2+\epsilon(\beta_1+\beta_2-1)}) = o(T^{-1/2+\epsilon_1})$$

Hence the series (45) is convergent as $T \to \infty$, indepedently of the β's.

§7. The problem of the order of magnitude of our singular series is by far much difficult, because the value of $\Delta^{2)}t^{\rho_1+\rho_2+1}$ depends on the relative value of t with respect to $|\rho_1+\rho_2|$.

As stated above, we have the trivial approximation

$$\Delta^{2)}t^{\rho_1+\rho_2+1} = o(t^{2\vartheta+1}) \tag{47}$$

for every ρ_1 and ρ_2.

Furthermore, due to the fact that

$$\Delta^{2)}t^{\rho_1+\rho_2+1} = \{t^{\rho_1+\rho_2+1} - (t-1)^{\rho_1+\rho_2+1}\}\{(t-1)^{\rho_1+\rho_2+1} - (t-2)^{\rho_1+\rho_2+1}\} =$$

$$= \int_{t-1}^{t} \frac{u^{\rho_1+\rho_2}}{\rho_1+\rho_2+1} du - \int_{t-2}^{t-1} \frac{u^{\rho_1+\rho_2}}{\rho_1+\rho_2+1} du$$

we have

$$| \Delta^{2)} t^{\rho_1+\rho_2+1} | < c_1 \frac{t^{2\vartheta}}{\gamma_1+\gamma_2} + c_2 \frac{t^{2\vartheta}}{\gamma_1+\gamma_2} < c_3 \frac{t^{2\vartheta}}{\gamma_1+\gamma_2} \tag{48}$$

valid also for every ρ_1 and ρ_2 .

Finally, in paragraph 14, later, we shall deduce that

$$\Delta^{2)} t^{\rho_1+\rho_2+1} \sim c_4(\gamma_1+\gamma_2)^2 t^{i(\gamma_1+\gamma_2)} \tag{49}$$

if $| \gamma_1 + \gamma_2 | < t$.

Now, if conjecture A of P.N.III is correct, we should have the bounds:

$$t < s(t) < e^{\gamma} t. \log\log t \tag{50}$$

It seems difficult to obtain so sharp approximations using the preceding considerations.

§8. We return now to the considerations of paragraph 4 b). Now we should find the branch points of $\mathcal{L}\{\psi(x)\}$ But it happens that it is much easy to determine those of the function $\mathcal{L}\{\vartheta(x)\}$ where $\vartheta(x) = \sum \log p$ and $p \leq x$, that is a function closely related to $\psi(x)$. (We have the known relation :

$$\psi(x) = \vartheta(x) + \vartheta(\sqrt{x}) + \vartheta(\sqrt[3]{x}) + \cdots \tag{51}$$

between them.)

Now holds that

$$\frac{\vartheta(x+0)+\vartheta(x-0)}{2} = \begin{cases} \sum\limits_{p\leq x} \log p & \text{if } x \neq p \\ \sum\limits_{p\leq x} \log p - \frac{1}{2}\log x & \text{if } x = p \end{cases} \tag{52}$$

We form after:

$$\Phi(x) = \frac{\vartheta(x+0)+\vartheta(x-0)}{2} - \frac{\vartheta(x-1+0)+\vartheta(x-1-0)}{2} \tag{53}$$

and find that

$$\Phi(x) = \begin{cases} \log x & \text{if } p < x < p+1 \\ \frac{1}{2}\log x & \text{if } x = p \\ 0 & \text{in any other case} \end{cases} \tag{54}$$

with an analoguous formula for $\Phi(t-x)$

We define again

$$I^*(x) = \Phi(x)\Phi(t-x) \tag{55}$$

and deduce that

$$I^*(x) = \begin{cases} i) \quad \log x . \log(t-x) & \text{if } p_1 < x < p_1 + 1 \\ & \text{and besides} \\ & p_2 < t-x < p_2+1 \\ ii) \quad \frac{1}{4}\log x \log(t-x) & \text{if } x = p_1, \quad t-x = p_2 \\ iii) \quad 0 & \text{in any other case} \end{cases} \tag{56}$$

Denoting now (with Hardy-Littlewood) with $\nu(t)$ to the function

$$\nu(t) = \sum_{p_1+p_2=t}\sum \log p_1 \log p_2 \tag{57}$$

we can evaluate it as an integral by use of (55) i):

$$\nu(t) = \int_1^{t-1} I^*(x)dx = \int_0^t \Phi(x)\Phi(t-x)dx \tag{58}$$

In fact, both $\Phi(x)$ and $\Phi(t-x)$ are discontinuous functions with horizontal steps of lenght unity, and the same property has then $I^*(x)$. Hence the value of $\nu(t)$ is the sum of the areas below the steps of $I^*(x)$.

From (57) follows:

$$\mathcal{L}\{\nu(t)\} = \mathcal{L}^2\{\Phi(x)\}$$

But (as in (24)) due to (52) we deduce that

$$\mathcal{L}\{\Phi(x)\} = (1-e^{-s})\mathcal{L}\{\vartheta(x)\} \tag{59}$$

or, what is the some thing:

$$\mathcal{L}\{\nu(t)\} = (1 - e^{-s})^2 \mathcal{L}^2\{\vartheta(x)\} \tag{60}$$

Hence:

$$\frac{\nu(t+0) + \nu(t-0)}{2} = \mathcal{L}^{-1}\{(1 - e^{-s})^2 \mathcal{L}^2\{\vartheta(x)\}\} \tag{61}$$

$$= \Delta^{2)} \mathcal{L}^{-1}\{\mathcal{L}^2\{\vartheta(x)\}\}$$

For the sake of simplicity, let us denote the left hand side as:

$$\nu^*(t) = \frac{\nu(t+0) + \nu(t-0)}{2} \tag{62}$$

§9. We are going now to prove that

$$\mathcal{L}\{\vartheta(x)\} = \frac{1}{s} \sum_p \log p e^{-ps} \tag{63}$$

in fact, we have that

$$\mathcal{L}^{-1}\{\frac{1}{s} \sum_p \log p e^{-ps}\} = \sum_p \log p \mathcal{L}^{-1}\{\frac{e^{-ps}}{s}\}$$

due to the uniform convergence of the left hand series for $\sigma = Re(s) > 0$.
But

$$\mathcal{L}^{-1}\{\frac{e^{-ps}}{s}\} = U(x - p) \tag{64}$$

where $U(x-p)$ is Heaviside's unitary function:

$$U(x-p) = \begin{cases} 0 & \text{if } x < p \\ 1 & \text{if } x > p \end{cases} \tag{65}$$

Is evident, then that

$$\mathcal{L}^{-1}\{\frac{1}{s} \sum_p \log p e^{-ps}\} = \sum_{p \leq x} \log p = \vartheta(x)$$

for non integral x, from which follows (63)
In change, the corresponding formula for $\psi(x)$ (that we shall not use) is:

$$\mathcal{L}\{\psi(x)\} = \frac{1}{s} \sum_p \log p \{e^{-sp} + e^{-sp^2} + e^{-sp^3} + \cdots\}$$

§10. The function that appears at the right hand side of (63) has the line $s = \sigma = 0$ as a natulal boundary.

According to formula (213) of ref.[5] (with an excessively short explanation) we have that

$$\sum_{p>2} \log p.x^p - \psi_\rho(x) < A_{20} n^{\vartheta+1/4+\epsilon} \tag{66}$$

" when are considered all the Farey fractions h with $0 < q \leq \sqrt{n}$ and $0 \leq \frac{h}{q} \leq 1$ " and it is assumed that $|\ x\ | = e^{-\frac{1}{n}}$.

In the preceding formula, ρ denotes a prime number, ϑ is the upper limit of the real part of the zeros of the L - functions, ρ is a primitive root of unity:

$$\rho = e^{2\pi i h/q} \text{ with } 0 \leq \frac{h}{q} \leq 1 \text{ and } (h,q) = 1 \tag{67}$$

and

$$\psi_\rho(x) = \sum_q \sum_\rho \psi(x) = \sum_q \sum_p \frac{\mu(q)}{\varphi(q)(1 - \frac{x}{\rho})} =$$

$$= \sum_q \sum_{\substack{h=0 \\ (h,q)=1}}^{q} \frac{\mu(q)}{\varphi(q)(1 - xe^{-2\pi i h/q})}$$

Here $\varphi(q)$ is Euler's function: quantity of numbers $\leq q$ and relatively prime with q, and $\mu(q)$ is Moebius's function.

Then formula (66), when written explicitly and developed is

$$\sum_{p>2} \log p.x^p = \sum_{q=1}^{[\sqrt{n}]} \sum_{h=0}^{q} \frac{\mu(q)}{\varphi(q)(1 - xe^{-2\pi i h/q})} + An^{\vartheta+1/4+\epsilon} \tag{68}$$

where (h,q) = 1.

The first sum sign denotes that is performed the Farey disection of orden $[\sqrt{n}]$, and the second summation is extended to all the q-th primitive roots of unity.

All this can be confirmed consulting the original paper (ref. [3], p.19-23).

Putting in (68) $x = e^{-s}$ (from which follows $\sigma = Re(s) = \frac{1}{n}$). we obtain:

$$\sum_{p>2} \log p.e^{-ps} = \sum_{q=1}^{[\sqrt{n}]} \sum_{h=0}^{q} \frac{\mu(q)}{\varphi(q)(1 - e^{-(s+2\pi i h/q)})} + An^{\vartheta+1/4+\epsilon} \tag{69}$$

(We shall not write in what follows the condition $(h,q) = 1$ in the second sum by typographycal case, but must be assumed in every case).

We separate now in the right hand side the terms with $h = 0$ and $h = q$. We have then:

$$\sum_{p>2} \log p . e^{-ps} = \sum_{q=1}^{[\sqrt{n}]} \frac{2\mu(q)}{\varphi(q)(1-e^{-s})} + \tag{70}$$

$$+ \sum_{q=1}^{[\sqrt{n}]} \sum_{h=1}^{q-1} \frac{\mu(q)}{\varphi(q)(1-e^{-(s+2\pi i h/q)})} + An^{\vartheta+1/4+\epsilon}$$

Here we must represent the denominators in the vicinity of $s = -2\pi ih/q$.
From the known expansion

$$\frac{1}{1-e^{-x}} = \frac{1}{x} + \frac{1}{2} + \frac{B_2}{2!}x - \frac{B_3}{3!}x^2 + \cdots \tag{71}$$

where the B_ν are the Bernoulli numbers, and $|\, x \,| < 2\pi$, we deduce:

$$\frac{1}{1-e^{-(s+2\pi ih/q)}} = \frac{1}{s+2\pi ih/q} + \frac{1}{2} + \frac{B_2}{2!}(s+2\pi ih/q) + \cdots \tag{72}$$

This is a convergent series for those values of s such that

$$|\, s + 2\pi ih/q \,| < 2\pi$$

As $\frac{h}{q} < 1$, always there are values of s with this property and such that $Re(s) = \sigma = \frac{1}{n}$.
In fact must hold:

$$|\, \frac{1}{n} + 2\pi ih/q \,| < 2\pi$$

that is to say

$$\frac{1}{n^2} + 4\pi\frac{h^2}{q^2} < 4\pi \text{ or } \frac{1}{n^2} + 4\pi\frac{(q-1)^2}{q^2} < 4\pi$$

from which follows

$$\frac{1}{n^2} < 4\pi(\frac{2}{q} - \frac{1}{q^2}) < 4\pi\frac{2}{q} < \frac{8\pi}{[\sqrt{n}]}$$

that actually holds true.
Then from (72) we deduce

$$|\, \frac{1}{2} + \frac{B_2}{2!}(s+2\pi ih/q) - \frac{B_3}{3!}(s+2\pi ih/q)^2 + \cdots \,| \leq M \tag{73}$$

because the series is convergent, and we can write

$$\frac{1}{1-e^{-(s+2\pi ih/q)}} = \frac{1}{s+2\pi ih/q} + M_1 \tag{74}$$

Replacing in (70) we obtain:

$$\sum_{p>2} \log p . e^{-ps} = \sum_{q=1}^{[\sqrt{n}]} \frac{2\mu(q)}{\varphi(q)(1-e^{-s})} + An^{\vartheta+1/4+\epsilon} + \tag{75}$$

$$+ \sum_{q=1}^{[\sqrt{n}]} \sum_{h=1}^{q-1} \frac{\mu(q)}{\varphi(q)(s+2\pi ih/q)} + \sum_{q=1}^{[\sqrt{n}]} \sum_{h=1}^{q-1} \frac{\mu(q)}{\varphi(q)} M_1$$

But

$$\left| \sum_{h=1}^{q-1} \frac{\mu(q)}{\varphi(q)} \right| < \frac{\varphi(q-1)}{\varphi(q)} < e^{\gamma} \log\log q \tag{76}$$

so that

$$\left| \sum_{q=1}^{[\sqrt{n}]} \sum_{h=1}^{q-1} \frac{\mu(q)}{\varphi(q)} M_1 \right| < \sum_{q=1}^{[\sqrt{n}]} e^{\gamma} \log\log q < A_1 \sqrt{n} \log\log n \tag{77}$$

Hence the last term of (75) can be included in the term $An^{\vartheta+1/4+\epsilon}$.
Summaryzing, (69) can be replaced by (75), and this by

$$\sum \log p . e^{-ps} = \sum_{q=1}^{[\sqrt{n}]} \frac{2\mu(q)}{\varphi(q)(1-e^{-s})} + An^{\vartheta+1/4+\epsilon} + \tag{78}$$

$$+ \sum_{q=1}^{[\sqrt{n}]} \sum_{h=1}^{q-1} \frac{\mu(q)}{\varphi(q)(s+2\pi ih/q)}$$

In the circle method n is the number whose decomposition in sum of primes is sought. In our method, it is a parameter whose value we shall choos at the end of the calculation.

Finally, due to (63), we deduce:

$$\mathcal{L}\{\vartheta(x)\} = \frac{1}{s} \frac{1}{(1-e^{-s})} F(n) + A \frac{n^{\vartheta+1/4+\epsilon}}{s} + \tag{79}$$

$$+ \sum_{q=1}^{[\sqrt{n}]} \sum_{h=1}^{q-1} \frac{\mu(q)}{\varphi(q) s (s+2\pi ih/q)}$$

with

$$F(n) = \sum_{q=1}^{[\sqrt{n}]} \frac{2\mu(q)}{\varphi(q)} < \frac{A_2}{\log n} \tag{80}$$

as will be shown in paragraph 24.

(to be continued)