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THE ANATOMY OF EVEN EXPONENT PYTHAGOREAN TRIPLES

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Abstract

This paper uses a novel approach to analyse the diophantine equation $d^{2n} = e^{2n} + f^{2n}$ in a number of ways for even and odd n, and with the use of the equivalence classes of the modular ring $\mathbb{Z}_6$, the characteristics of primitive Pythagorean triples and right-most digit patterns.

1. Introduction

We have previously used the modular ring $\mathbb{Z}_6$ to classify Pythagorean triples (Pts) in a novel way [4]. This ring is used, together with a z-j grid that accommodates the primitive Pythagorean triples (pPts) [5], to analyse in a different approach why exponents higher than 2 are excluded from the integer triple solutions in $\mathbb{Z}_6$. Of course the unrestricted case has been the focus of recent successful attention [6]. The quest to make the solution as accessible as the problem remains undiminished.

If c, b and a are components of a Pythagorean triple, with $c > b > a$, and $c = d^n$, $b = e^n$ and $a = f^n$, with n a positive integer, then

$$d^{2n} = e^{2n} + f^{2n} \tag{1.1}$$

is not valid when $n > 1$, which means that the components of a pPt cannot all be equal to integers raised to the same power. For example, in the case where $n = 2$, c, b and a cannot all be perfect squares for a given pPt.

The basic equations for the z-j grid are [5]:

(a) z Odd:

$$c = j^2 + \left(j + z^{\frac{1}{2}}\right)^2 \tag{1.2}$$

$$b = c - z \tag{1.3}$$

$$a = c - 2j^2 \tag{1.4}$$

where $z = c - b = (2t-1)^2$, $t = 1,2,3,...$, and $j > \left(\frac{z}{2}\right)^{\frac{1}{2}}$, $j = 1,2,3,....$,

(b) z Even:

$$c = \left(\frac{z}{2}\right) + \left(\left(\frac{z}{2}\right)^{\frac{1}{2}} + (2j-1)\right)^2 \tag{1.5}$$

$$b = c - z \tag{1.6}$$

$$a = c - (2j-1)^2 \tag{1.7}$$

where $z = 2t^2$, $t = 1,2,3.....$, and $j > \frac{\left(z^{\frac{1}{2}}+1\right)}{2}$, $j = 1,2,3....$

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Table 1.1: Plan of paper

For both (a) and (b) we shall consider two cases of equation (1.1), in sections 2 and 3, namely, (2) n even, with $c = d^n$, $b = e^n$ and $a = f^n$, $n = 2,4,6,...$, (3) n odd, with

$c = d^n$, $b = e^n$ and $a = f^n$, $n = 1, 3, 5, \ldots$ The plan of the paper is set out in Table 1.1, in which FLT stands for Fermat's Little Theorem.

2. Case 1: n even

As shown previously [4] squares of even integers only occur in equivalence classes $\overline{1}$ and $\overline{3}$ of $\mathbb{Z}_6$. Whereas squares of odd integers fall only in classes $\overline{4}$ and $\overline{6}$. Because of this and the characteristics of $\mathbb{Z}_6$, c (which is always odd) must fall in class $\overline{4}$. When $(c - b)$ or z is even, b is odd and falls in class $\overline{4}$ as well. The remaining component, a, falls in class $\overline{3}$. When z is odd the classes for b and a are reversed, so that the three components are confined to the classes $\overline{4}\ \overline{4}\ \overline{3}$ (even z) and $\overline{4}\ \overline{3}\ \overline{4}$ (odd z). This applies for all even values of n.

2 (a) z Odd

When n is even the three components are confined to the classes $\overline{4}\ \overline{3}\ \overline{4}$. Apart from this constraint, the characteristics of the z-j grid [5] and the required right-most digits for even powered integers prevent equation (1.1) from having an integer solution. For $\overline{4}, \overline{3}, \overline{4}$, $(j, 3) > 1$ but $(z, 3) = 1$ [4,5].

(2a.1) Grid-grid Method. With the components c and b both squares and since their difference equals z, which is a square, there will be two 'intertwined' triples.

Let $d^2 - e^2 = z_2$ and $d - e = z_1$. Then, for the $\left(d, e, z_2^{\frac{1}{2}}\right)$ triple, (section 1):

$$d = j_1^2 + \left(j_1 + z_1^{\frac{1}{2}}\right)^2 \qquad (2.1)$$

$$e = d - z_1 \qquad (2.2)$$

$$z_2^{\frac{1}{2}} = d - 2j_1^2 \qquad (2.3)$$

Whilst for the (d^2, e^2, f^2)triple, i.e. (c, b, a) :

$$d^2 = j_2^2 + \left(j_2 + z_2^{\frac{1}{2}}\right)^2 \qquad (2.4)$$

$$e^2 = d^2 - z_2 \qquad (2.5)$$

$$f^2 = d^2 - 2j_2^2 \qquad (2.6)$$

From equation (2.3) we get

$$d^2 = 4j_1^4 + z_2 + 4j_i^2 z_2^{\frac{1}{2}} \tag{2.7}$$

so that by, combining equations (2.4) and (2.7) we obtain

$$2j_1^2\left(j_1^2 + z_2^{\frac{1}{2}}\right) = j_2\left(j_2 + z_2^{\frac{1}{2}}\right) \tag{2.8}$$

If $j_1^2 > j_2$ then $2j_1^2 < j_2$, which is impossible, so that $j_2 > j_1^2$ and $j_2 < 2j_1^2$. Hence

$$j_1^2 < j_2 < 2j_1^2 \tag{2.9}$$

From equation (2.8) we have

$$z_2^{\frac{1}{2}} = \left(j_1^2 - j_2\right)\frac{\left(j_1^2 + j_2\right)}{\left(j_2 - 2j_1^2\right)} + \frac{j_1^4}{\left(j_2 - 2j_1^2\right)} \tag{2.10}$$

Because (a) the first term on the RHS of equation (2.10) is smaller then the second, and (b) $\left(j_2 - 2j_1^2\right)$ and $\left(j_1^2 - j_2\right)$ are both negative, the first term is positive, whereas the second term is negative. Thus $z_2^{\frac{1}{2}}$ is negative.

Let $z^{\frac{1}{2}} = -z'^{\frac{1}{2}}$ so that $z = z'$. Then

$$e^2 = 2j^2 - 2jz'^{\frac{1}{2}} \tag{2.11}$$

and

$$f^2 = -2jz'^{\frac{1}{2}} + z' \tag{2.12}$$

If $j > z'^{\frac{1}{2}}$, then f^2 will be negative and have imaginary roots. However, if $z'^{\frac{1}{2}} > j$, e^2 will be negative. Thus d, e and f cannot all simultaneously be integers.

It should be noted that if we simply square c^2 to get $c^4 = (bc)^2 + (ac)^2$, (which of course will not be a primitive Pt), it might appear that c does not have to be a perfect square and only (bc) and (ac) need be squares, say g^2 and h^2 respectively. However, since $h = \left(\frac{a}{b}\right)^{\frac{1}{2}} g$ and taking $(c - a) = 2j_1^2$ (from the (c, b, a) triple) and $\left(c^2 - ac\right) = 2j_2^2$ (for the $\left(c^2, bc, ac\right)$triple) so that $c^{\frac{1}{2}} j_1 = j_2$, we see that we need to have c, b and a perfect squares for an integer solution, as above.

Another situation to note is that if $\left(j + z^{\frac{1}{2}}\right)$ and j of a pPt (c', b', a') are equal to b and a, respectively, of another pPt (c, b, a), then c' will be a square. Alternatively, if c and b are taken for $j + z^{\frac{1}{2}}$ and j, then the new pPt gives a' as a square. These are two grid-grid cases that lead to the same conclusion as above, namely, that all the components cannot be squares. In the first case, using equation (1.2) to (1.4) we can easily show that $c' = \left(2j_1^2 + 2j_1 z_1^{\frac{1}{2}} + z_1\right)^2$ where the subscript indicates that j and $z^{\frac{1}{2}}$ are from the (c, b, a) triple. Thus c' is a square. However,

$$b' = 4j_1 z_1^{\frac{1}{2}}\left(3j_1 z_1^{\frac{1}{2}} + z_1 + 2j_1^2\right)$$

and

$$a' = z_1^{\frac{1}{2}}\left(8j_1^3 - 4j_1 z_1 - z_1^{\frac{3}{2}}\right) + 4j_1^4.$$

For example, if (c, b, a)=$(5, 4, 3)$ then $z = 1$, $j = 1$ so that $c' = 25$, b$' = 24$ and $a' = 7$.

A similar result is obtained for the second grid-grid case, where

$$a' = \left(2j_1 z_1^{\frac{1}{2}} + z_1\right)^2$$

and

$$b' = 4j_1\left(2j_1^3 + z_1^{\frac{3}{2}} + 3j_1 z_1 + 4j_1^2 z_1^{\frac{1}{2}}\right)$$

and

$$c' = 8j_1^4 + z_1^{\frac{1}{2}}\left(12j_1^2 z_1^{\frac{1}{2}} + 4j_1 z_1 + 16j_1^3 + z_1^{\frac{3}{2}}\right).$$

So with the same (c, b, a) as before, we get $(c', b', a') = (41, 40, 9)$.

(2a.2) Fermat's "Little Theorem" and Right-Most Digit Method.
Since

$$(c-b)(c+b)=a^2 \tag{2.13}$$

and $(c-b)$ or z is a square, $(c+b)$ must also be a square. As well, with the components all squares, i.e. $d^2, e^2 f^2$, the right-most digits (indicated by an *) of c, $(c-b)$ and $(c+b)$ must be $1, 5$ or 9 and the right-most digits of b must be $0, 4$ or 6.

According to Fermat's "little theorem" [1], for prime p, $(n,p)=1$, and $k_N \epsilon \mathbb{Z}$.

$$N^{p-1}=1+pk_N \tag{2.14}$$

Thus, if c and b are prime to 5,

$$c^2=d^4=1+5k_d \tag{2.15}$$

and

$$b^2=e^4=1+5k_e \tag{2.16}$$

However for an integer triple, a cannot also be prime to 5 [4].

Alternatively, c and a could be prime to 5 and b not prime to 5.

Hence there are two cases to consider.

(i) c and b prime to 5.
As can be seen from Table 2.1 $(c-b)$ and $(c+b)$ cannot simultaneously be squares as the right-most digits cannot be 3 or 7 for squares.

c^* / b^*	1	9	
4	7	5	$(c-b)^*$
	5	3	$(c+b)^*$
6	5	3	$(c-b)^*$
	7	5	$(c+b)^*$

Table 2.1

c^* / b^*	1	9	
0	1	9	$(c-b)^*$
	1	9	$(c+b)^*$
4	7	5	$(c-b)^*$
	5	3	$(c+b)^*$
6	5	3	$(c-b)^*$
	7	5	$(c+b)^*$

Table 2.2

(ii) c and a prime to 5.

This gives $(d^2)* = c* = 1$ or 9, whilst $(e^2)* = b* = 0, 4$ or 6. As can be seen from Table 2.2, when $c* = 1$ and $b* = 0$ or $c* = 9$ and $b* = 0$, squares are possible for both $(c - b)$ and $(c + b)$.

However, if $(d^2)* = c* = 9$ then $d* = 7$ or 3, and since $(e^2)* = b* = 0$, $e* = 0$. This gives $(d - e)* = 7$ or 3, digits that are invalid as $(d - e)$ is a square (section 2a.1). Thus, $(d^2)* = 1$ only, so that $d* = 1$ or 9. As well, with $(e^2)* = 0$, $z* = 1$. It follows that $(f^2)* = 1$. Thus, since d^2, f^2 and z are all odd squares, and prime to 3, d, f and $z^{\frac{1}{2}}$ must follow

$$x = A_x + 30t_x \tag{2.17}$$

Where $A_x = 19$ or 31 when x is in class $\overline{4}$ or $A_x = 29$ or 41 when x is in class $\overline{2}$. Since $e* = 0$

$$e = B + 60p \tag{2.18}$$

where t_x and p are integers.

With $B = 0$ or 30, and $(d^2 - e^2) = z$,

$$A_d^2 + 30^2 t_d^2 + 60 A_d t_d - B^2 - 60^2 p^2 - 120 Bp = A_z^2 + 30^2 t_z^2 + 60 A_z t_z \tag{2.19}$$

When $B = 30$, division by 120 gives a residual of 15/2 so that no integer solution occurs. When $B = 0$, it is more useful to use the relationship $(d^2 - f^2) = 2j^2$ with d and f from equation (2.17) and then it is clear that j must be even and fall in class $\overline{3}$ and $j* = 0$.

Hence

$$j = D + 60s \tag{2.20}$$

With $s = 0, 1, 2, 3...$and $D = 30$ or 60.

Using equation (1.3), i.e. $e^2 = \left(2jz^{\frac{1}{2}} + 2j^2\right)$,

$$60^2 p^2 = 2D^2 + 4 \times 60Ds + 2 \times 60^2 s^2 + 2DA_z + 60Dt_z + 120A_z s + 120 \times 30t_z s \tag{2.21}$$

when $D = 30$, division of equation (2.21) by 120 gives a non-integer residual of $(2 \times 30^2 + 60A_z)/120$. The same applies for $D = 60$ when s is even (division by 240 gives a non-integer residual).

Equation (2.21) may be put in the form

$$2(1+s)^2 + \left(t_z + \left(\frac{A_z}{30}\right)\right)(1+s) - p^2 = 0 \tag{2.22}$$

so that

$$(1+s) = \frac{\left(-\left(t_z + \left(\frac{A_z}{30}\right)\right) + \left(\left(t_z + \left(\frac{A_z}{30}\right)\right)^2 + 8p^2\right)^{\frac{1}{2}}\right)}{4} \tag{2.23}$$

As A_z is prime, s cannot be an integer whatever parity.

Let $\varepsilon = A_z/30$ (values being $0.63\ddot{3}$, $0.96\ddot{6}$, $1.03\ddot{3}$ or $1.36\ddot{6}$). For an integer solution for equation (2.23) it is necessary that $(t_z^2 + \varepsilon^2 + 2\,\varepsilon\, t_z + 8p^2)^{\frac{1}{2}} = E + \varepsilon$, where E is an integer, and that $(E - t_z)/4 = I$, where I is an integer. Combining these restrictions gives: $I = (E - F + \varepsilon - \delta)/2$ with $(F + \delta) = p^2/I$, $0 \leq \delta < 1$. If p or δ are zero, I cannot be an integer. Otherwise, $(\varepsilon - \delta + 1)/2$ or $(\varepsilon - \delta)/2$ would have to be integer. But $(\varepsilon - \delta) < 1$ for $A_z^* = 9$. When $A_z^* = 1$, the fractions become $1 + (\varepsilon' - \delta)/2$ or $(\varepsilon' - \delta + 1)/2$ with $\varepsilon' = 0.03\ddot{3}$ or $0.36\ddot{6}$, so that the same result as above applies. If $\varepsilon = \delta$, then E, F and t_z have the same parity.

If $\delta = \varepsilon$, then $(F + \varepsilon) = p^2/(1+s) = e^2/60j$, since $B = 0$, $D = 60$. Then $(e^2/2j) = 30F + A_z$. But $j^* = e^* = 0$ with $j < e$, and with $e = 6r_3$, $r_3^* = 0$ or 5, so that $(18r_3^2/j) = 30F + A_z$. However, $A_z{}^* = 9$ only applies so that $\delta \neq \varepsilon$.

2(b): z Even

The analysis for this parity of z follows that given for the odd z.

(2b.1) Grid-grid Method. As for z odd, with the components all squares, two triples arise involving two of the components, that is, (d^2, e^2, f^2) and $(d, (2j-1), f)$.

Using equations (1.5) to (1.7), the components of the second triple are given by:

$$d = \left(\left(\frac{z_1}{2}\right)^{\frac{1}{2}}\right)^2 + \left(\left(\frac{z_1}{2}\right)^{\frac{1}{2}} + (2j_1 - 1)\right)^2 \tag{2.24}$$

$$(2j_2 - 1) = d - z_1 \tag{2.25}$$

$$f = d - (2j_1 - 1)^2 \tag{2.26}$$

From equation (2.25)

$$d^2 = z_1^2 + 2z_1(2j_2 - 1) + (2j_2 - 1)^2 \tag{2.27}$$

which, combined with equation (1.5) for d^2 and with $z_2 = z$, gives

$$\left(\frac{z_2}{2}\right)^{\frac{1}{2}}\left(2\left(\frac{z_2}{2}\right)^{\frac{1}{2}} + 2(2j_2 - 1)\right) = z_1(z_1 + 2(2j_2 - 1)) \tag{2.28}$$

which show that

$$\left(\frac{z_2}{2}\right)^{\frac{1}{2}} < z_1 < 2\left(\frac{z_2}{2}\right)^{\frac{1}{2}} \tag{2.29}$$

Equation (2.28) may be expressed

$$(2j_2 - 1) = \left(z_1 - \left(\frac{z_2}{2}\right)^{\frac{1}{2}}\right)\left(z_1 + \left(\frac{z_2}{2}\right)^{\frac{1}{2}}\right) / \left[2\left(\left(\frac{z_2}{2}\right)^{\frac{1}{2}} - z_1\right)\right]$$

$$- \left(\frac{z_2}{2}\right) / \left[2\left(\left(\frac{z_2}{2}\right)^{\frac{1}{2}} - z_1\right)\right] \tag{2.30}$$

The first term is negative and larger then the second (which is positive) so that $(2j_2 - 1)$ is negative.

As in the case of the odd z result, equation (2.27) is not of the right form.

Let $(2j-1) = -q$, then

$$e^2 = -2\left(\frac{z}{2}\right)^{\frac{1}{2}} q + q^2 \tag{2.31}$$

and

$$f^2 = z - 2\left(\frac{z}{2}\right)^{\frac{1}{2}} q \tag{2.32}$$

If $(z/2)^{\frac{1}{2}} > q$, then e^2 is negative. But if $q > (z/2)^{\frac{1}{2}}$ then $2q(z/2)^{\frac{1}{2}} > z$ so f^2 is negative and no integer solutions are possible for d, e and f simultaneously.

(2b.2) Fermat's "little theorem" and right-most digit method.

From equations (1.5) and (1.7)

$$(c-a) = (2j-1)^2 \tag{2.33}$$

and

$$(c+a) = \left(2\left(\frac{z}{2}\right)^{\frac{1}{2}} + 2j - 1\right)^2 \tag{2.34}$$

As for z odd, there are two cases to consider.

(i): c and a prime to 5.

Since c (odd) and a (even) are both squares, $c^* = 1$ or 9 and $a^* = 4$ or 6.

As can be seen from Table 2.3, $(c-a)$ and $(c+a)$ cannot simultaneously be squares as required so that the components cannot all be squares in this case.

(ii): c and b prime to 5.

This means that $a^* = 0$, 4 or 6. Table 2.4 indicates that $(c-a)$ and $(c+a)$ can both be squares if $c^* = 1$ or 9 and $a^* = 0$. As $(2j-1)^2$ is odd the right-most digits will be 1, 5 or 9 and with a^* or $(f^2)^* = 0$, $f^* = 0$, then $d^* = 1$ or 9 only, so that $(d^2)^* = 1$ only, and $z^* = 0$.

A similar result as for odd z follows.

a^* \ c^*	1	9	
4	7	5	$(c-a)^*$
	5	3	$(c+a)^*$
6	5	3	$(c-a)^*$
	7	5	$(c+a)^*$

Table 2.3

a^* \ c^*	1	9	
0	1	9	$(c-a)^*$
	1	9	$(c+a)^*$
4	7	5	$(c-a)^*$
	5	3	$(c+a)^*$
6	5	3	$(c-a)^*$
	7	5	$(c+a)^*$

Table 2.4

(2.3) Fermat's Theorem Method. Another simple analysis of the 4th power triples is based on a theorem of Fermat [2]. This states: the area of a right-angled triangle with integers for sides can never be the square of an integer, i.e.

$$\frac{1}{2}ab \neq M^2 \tag{2.35}$$

where M is an integer. With $b = 2mn$, $a = m^2 - n^2$ and taking $m = u^2$ and $n = v^2$, the area becomes

$$mn(m^2 - n^2) = (uv)^2 \left(u^4 - v^4\right) \tag{2.36}$$

If $u^4 - v^4 = w^2$, then the area $= (uvw)^2$. But this is impossible so that $u^4 - v^4$ is not equal to a square [2]. Consequently, with $c = d^2$, $b = e^2$ and $a = f^2$ and since $d^4 - e^4 \neq a^2$, then $d^4 - e^4 \neq f^4$.

Since, when n is even, equation (1.1) can be put in the form of a four power triple, such triples cannot have integer solutions.

3. Case 2: n Odd

Since integers and their odd power exist in the same equivalence class, four classes will be involved in this case, for each parity of z (Table 1.1).

3(a): z Odd

(i) Class set $\overline{2}\ \overline{1}\ \overline{6}$

To illustrate the method $n = 3$ is used. Extension to the higher prime exponents is then easily made.

For the class set $\overline{2}\ \overline{1}\ \overline{6}$ the triple components c, b and a will be cubes; $c = d^3$, $b = e^3$ and $a = f^3$. The class equations for d, e and f will be [4]

$$d = 6r_2 - 1 \tag{3.1}$$
$$e = 6r_1 - 2 \tag{3.2}$$
$$f = 6r_6 + 3 \tag{3.3}$$

where r is the row for the subscripted class. Hence

$$d^3 = 6(36r_2^3 - 18r_2^2 + 3r_2) - 1 \tag{3.4}$$
$$e^3 = 6(36r_1^3 - 36r_1^2 + 12r_1 - 1) - 2 \tag{3.5}$$
$$f^3 = 6(36r_6^3 + 54r_6^2 + 27r_6 + 4) + 3 \tag{3.6}$$

For numbers raised to a power, we use a tilde to distinguish the r values. Hence, from equations (3.4) to (3.6)

$$d^3 = 6\tilde{r}_2 - 1 \tag{3.7}$$
$$e^3 = 6\tilde{r}_1 - 2 \tag{3.8}$$
$$f^3 = 6\tilde{r}_6 + 3 \tag{3.9}$$

so that

$$\tilde{r}_2 = 36r_2^3 - 18r_2^2 + 3r_2 \tag{3.10}$$
$$\tilde{r}_1 = 36r_1^3 - 36r_1^2 + 12r_1 - 1 \tag{3.11}$$
$$\tilde{r}_6 = 36r_6^3 + 54r_6^2 + 27r_6 + 4 \tag{3.12}$$

Combining equations (1.2) to (1.4) and equations (3.7) to (3.9) gives

$$\tilde{r}_2 = \tilde{r}_1 + \frac{(z-1)}{6} \tag{3.13}$$
$$z\tilde{r}_1 = 3\tilde{r}_6^2 + 3\tilde{r}_6 + R \tag{3.14}$$

with $R = \dfrac{(-z^2 + 4z + 9)}{12}$

when $z = 1$,

$$\tilde{r}_2 = \tilde{r}_1 \tag{3.15}$$

Equating equations (3.10) and (3.11) and dividing by 3 gives a residual of $-\left(\frac{1}{3}\right)$ so that an integer solution is not possible for this z.

In general

$$\begin{aligned}\tilde{r}_2 - \tilde{r}_1 &= 3\big(12(r_2^3 - r_1^3) - 6(r_2^2 - 2r_1^2) + (r_2 - 4r_1)\big) + 1 \\ &= \frac{(z-1)}{6}\end{aligned} \tag{3.16}$$

Thus when $\dfrac{(z-7)}{18}$ is a non-integer there will be no integer solution. Substituting values of z into this function shows that for both r_2 and r_1 to be integers,

$$z^{\frac{1}{2}} = Z + 18k_z \tag{3.17}$$

where $Z = 5$ (class $\overline{2}$) or 13 (class $\overline{4}$).

Substitution of equation (3.17) into equation (3.14) and using equations (3.11) and (3.12) does not readily show non-integer characteristics.

If the modular ring $\mathbb{Z}_6$ is combined with the z-j equations in an alternative way, we get

$$2zd^3 = f^6 + z^2 \tag{3.18}$$
$$z = 6\tilde{r}_4 + 1 \tag{3.19}$$

and the classes for z given in Table 3.1. As can be seen, equation (3.17) is derived by this method as well (Table 3.1, Z values) The same type of analysis for j gives

$$4j^2 f^3 = e^6 - 4j^4 \tag{3.20}$$
$$j^2 = 6\tilde{r}_4' + 1 \tag{3.21}$$
$$j = J + 18k_j \tag{3.22}$$

where J values and classes for j are given in Table 3.1. For the components of the triple all to be cubes with integer roots, then the k_j and k_z values must be the same and d, e and f must be integer.

However, it is found that for common k_j and k_z values only one of the components has an integer cube root. Some cases are illustrated in Figures 1, 4 where $k_j - k_z$ values are plotted as decreasing values. The disparate curves are separated, crossing at non-integer N. For the sample shown (components up to 10^{10} used) the three curves never intersect simultaneously.

N_0	j	j^2	$z^{\frac{1}{2}}$	z	Z	J
1	$\overline{4}$	$\overline{4}$	$\overline{4}$	$\overline{4}$	13	7
2	$\overline{2}$	$\overline{4}$	$\overline{2}$	$\overline{4}$	5	11
3	$\overline{1}$	$\overline{1}$	$\overline{4}$	$\overline{4}$	13	16
4	$\overline{5}$	$\overline{1}$	$\overline{2}$	$\overline{4}$	5	20

Table 3.1:

Classes for j and z functions of set $\overline{2}\ \overline{1}\ \overline{6}$ and parameters for equations (3.17) and (3.22).

The numbers to form six power triples must exclude all numbers involved in a Pt, unless all are cubes. As well, if a number contains factors $p_1 p_2 p_3 ...$ it is only necessary that one of the components of a Pt is one of these factors, say p_1. Then the Pt can be multiplied throughout by $(p_2 p_3 ...)^2$ so that a Pt is formed containing that number. This gives a very large range of excluded numbers.

The above results indicate that only one component can be a cube. This is of interest since if we take a component of a Pt and raise it to the sixth power it can never equal the sum of two numbers, each raised to the sixth power if one of the components is not a cube. For example, if d, e and f are components of a Pt and f is raised to the sixth power, then

$$f^6 = d^2 f^4 - e^2 f^4 \tag{3.23}$$

so that, if $d^2 f^4 = g^6$ and $e^2 f^4 = h^6$,

$$\frac{d^2}{e^2} = \frac{g^6}{h^6} \tag{3.24}$$

so that $h = \left(\frac{e}{d}\right)^{\frac{1}{3}} g$ with $\frac{e}{d} < 1$. If either e or d is not a cube h will be irrational. Of course, the class set for equation (3.23) will be $\overline{6}\ \overline{6}\ \overline{3}$ which is not a pPt. Here there is no obvious indication that only one component can be a cube, namely f.

(ii) Class set $\overline{2}\ \overline{5}\ \overline{6}$

The class equations for d and f of this set will be equations (3.1) and (3.3) respectively. The equation for e, in class $\overline{5}$, is:

$$e = 6r_5 + 2 \tag{3.25}$$

so that, using the same method as for set $\overline{2}\ \overline{1}\ \overline{6}$, we get, for the rows of the cubes:

$$\tilde{r}_2 = 36r_2^3 - 18r_2^2 + 3r_2 \tag{3.26}$$
$$\tilde{r}_5 = 36r_5^3 + 36r_5^2 + 12r_5 + 1 \tag{3.27}$$
$$\tilde{r}_6 = 36r_6^3 + 54r_6^2 + 27r_6 + 4 \tag{3.28}$$

As for the set $\overline{2}\ \overline{1}\ \overline{6}$, we can derive

$$\tilde{r}_2 = \tilde{r}_5 + \tfrac{(z+3)}{6} \tag{3.29}$$
$$z\tilde{r}_5 = 3\tilde{r}_6^2 + 3\tilde{r}_6 + R \tag{3.30}$$

with $R = \big(-z^2 - 4z + 9\big)/12$; $\tilde{r}_5 > \tilde{r}_6$.

An integer solution of equation (3.29) requires that z is never prime to 3, so that

$$z^{\frac{1}{2}} = 3 + 6k_z \tag{3.31}$$

Alternatively, the ring properties of $d^3 - e^3 = z$ indicate that z falls in class $\overline{6}$, so that

$$z = 6\tilde{r}'_6 + 3 \tag{3.32}$$

Since $z^{\frac{1}{2}}$ also falls in class $\overline{6}$, $\tilde{r}'_6 = \left(6{r'_6}^2 + 6r'_6 + 1\right)$ and must therefore be odd. This means $\tilde{r}'_6$ falls in classes $\overline{2}$, $\overline{4}$ or $\overline{6}$. Substituting d, f and z from the above equations, into equation (3.18) and dividing by 36 gives a residual of $(62 + \tilde{r}'_6)/3$. Class $\overline{4}$ gives the only integer solution with $\tilde{r}'_6 = 6r'_4 + 1$, so that

$$z = 36r'_4 + 9 \tag{3.33}$$

which for integer solutions gives equation (3.31) as before, with $r'_4 = (k_z^2 + k_z)$.

Since $(d^3 - f^3) = 2j^2$ and d and f are in the same classes as the the previous set $\overline{2}\ \overline{1}\ \overline{6}$, odd j^2 is in class $\overline{4}$ and even j is in class $\overline{1}$.

With $j^2 = (6\tilde{r}'_4 + 1)$ substituted into equation (3.20) and dividing by 36 we get a residual of $-48(\tilde{r}'_4 + 1)/36$ which indicates $\tilde{r}'_4$ must fall in class $\overline{5}$. Hence the same two functions apply for odd j as shown in Table 3.1. For even j, $j^2 = 6\tilde{r}'_1 - 2$, and following the same procedure as for odd j, we get a residual of $(8\tilde{r}'_1 + 22)/3$ which gives an integer result only when $\tilde{r}'_1$ is in class $\overline{4}$. Again, the same functions of j apply as shown in Table 3.1.

As for class set $\overline{2}\ \overline{1}\ \overline{6}$, no common k_j, k_z are found for integer e, d or f. Hence the components cannot simultaneously all be cubes

(iii) Class set $\overline{4}\ \overline{3}\ \overline{2}$

For this set

$$d = 6r_4 + 1 \tag{3.34}$$

$$e = 6r_3 \tag{3.35}$$

$$f = 6r_2 - 1 \tag{3.36}$$

Since $c^2 - a^2 = b^2$, then

$$d^6 - f^6 = e^6 \tag{3.37}$$

Combining equations (3.34) to (3.37) and dividing throughout by 36, gives

$$6^4(r_4^6 - r_2^6) + 6^4(r_4^5 + r_2^5) + 15 \times 36(r_4^4 - r_2^4) + 20 \times 6(r_4^3 + r_2^3) + 15(r_4^2 - r_2^2) + (r_4 + r_2) = 6^4 r_3^6 \tag{3.38}$$

which, by factorising in the manner of Kummer [3] yields:

$$6^4(r_4^5 - r_4^4 r_2 + r_4^3 r_2^2 - r_4^2 r_2^3 + r_4 r_2^4 - r_2^5) + 6^4(r_4^4 - r_4^3 r_2 + r_4^2 r_2^2 - r_4 r_2^3 + r_2^4) + 15 \times 36(r_4^3 - r_4^2 r_2 + r_4 r_2^2 - r_2^3) + 20 \times 6(r_4^2 - r_4 r_2 + r_2^2) + 15(r_4 - r_2) = \left(\frac{6^4 r_3^6}{(r_2 + r_4)}\right) - 1 \tag{3.39}$$

If $(6^4 r_3^6/(r_2+r_4))$ is an integer then dividing equation (3.39) by 3 shows a non-integer relationship. However, if $[6^4 r_3^6/(r_2+r_4)]$ is non-integer, then since $[6^4 r_3^6 \gg (r_2+r_4)]$ we can let the non-integer be $(6A+e)$ where e is the decimal residual; $e<1$ and A an integer.

Then dividing by 3 gives a non-integer equation since $(1-e)/3$ will be less than 1.

The same applies in the general case with n even and m odd since the factorisation of $(r_4^n - r_2^n)$ and $(r_4^m + r_2^m)$ gives the same form of equations.

(iv) Class set $\overline{4}\ \overline{3}\ \overline{4}$

For this set

$$d = 6r_4 + 1 \tag{3.40}$$
$$e = 6r_3 \tag{3.41}$$
$$f = 6r_4' + 1 \tag{3.42}$$

Substituting these equations into equation (3.37) and dividing by 36, gives:

$$\begin{aligned} &6^4\left(r_4^6 - r_4'^6\right) + 6^4\left(r_4^5 - r_4'^5\right) + 15\times 36\left(r_4^4 - r_4'^4\right) \\ &+ 20\times 6\left(r_4^3 - r_4'^3\right) + 15\left(r_4^2 - r_4'^2\right) + \left(r_4 - r_4'\right) = 6^4 r_3^6 \end{aligned} \tag{3.43}$$

Here we can use

$$r_4^n - r_4'^n = \left(r_4 - r_4'\right)\left(r_4^{n-1} + r_4^{n-2} r_4' + r_4^{n-3} r_4'^2 + \ldots + r_4'^{n-1}\right) \tag{3.44}$$

which applies to all n.

Similar to equation (3.39) we have on the RHS

$$\left(\frac{6^4 r_3^6}{\left(r_4 - r_4'\right)}\right) - 1 \tag{3.45}$$

and the same arguments lead to the conclusion that a non-integer solution occurs for equation (3.43).

3(b): z even

The same type of analysis as used for z odd can be made.

(i) Class set $\overline{2}\ \overline{6}\ \overline{1}$

The same class equations as used for the set $\overline{2}\ \overline{1}\ \overline{6}$ apply (equations (3.1) to (3.3)) only the equations for e and f are interchanged.

Since $z = 2t^2$, the z, j functions (1.5) to (1.7) may be expressed, for convenience, with t replacing $(z/2)^{\frac{1}{2}}$. Then, using the same techniques as for z odd, we find

$$4t^2e^3 = f^6 - et^4 \tag{3.46}$$

and

$$2(2j-1)^2f^3 = e^6 - (2j-1)^4 \tag{3.47}$$

Using the rules of the modular ring and equations (3.46) and (3.47) we obtain Table 3.2 and the equations

$$t = Z' + 18k_z \tag{3.48}$$

$$(2j-1) = J' + 18k_j \tag{3.49}$$

As can be seen, by comparing the two sets of equations (1.2) to (1.4) and (1.5) to (1.7), $(z/2)^{\frac{1}{2}}$ or t replaces j and $(2j-1)^2$ replaces z in the latter, with e^3 and f^3 interchanged. This explains the values of the coefficients Z' and J' in Table 3.2.

Consequently, the same result is obtained in regard to integer values of d, e and f as for the odd z case.

No	$(2j-1)$	$(2j-1)^2$	t	t^2	Z'	J'
1	$\overline{2}$	$\overline{4}$	$\overline{4}$	$\overline{4}$	7	5
2	$\overline{2}$	$\overline{4}$	$\overline{2}$	$\overline{4}$	11	5
3	$\overline{4}$	$\overline{4}$	$\overline{1}$	$\overline{1}$	16	13
4	$\overline{4}$	$\overline{4}$	$\overline{5}$	$\overline{1}$	20	13

Table 3.2: Classes for the t (or $z/2^{\frac{1}{2}}$) and $(2j-1)$ functions and parameters for equations (3.48) and (3.49).

(ii) Class set $\overline{2}\ \overline{6}\ \overline{5}$

The arguments outlined for the class set $\overline{2}\ \overline{6}\ \overline{1}$ can be applied here, together with the results for the set $\overline{2}\ \overline{5}\ \overline{6}$ above.

(iii) Class set $\overline{4}\ \overline{2}\ \overline{3}$

Here

$$d = 6r_4 + 1 \tag{3.50}$$
$$e = 6r_2 - 1 \tag{3.51}$$
$$f = 6r_3 \tag{3.52}$$

Thus, by taking

$$d^6 - e^6 = f^6 \tag{3.53}$$

The same arguments as for the set $\overline{4}\ \overline{3}\ \overline{2}$ apply and equation (3.53) is non-integer.

(iv) Class set $\overline{4}\ \overline{4}\ \overline{3}$

Here

$$d = 6r_4 + 1 \tag{3.54}$$
$$e = 6r_4' + 1 \tag{3.55}$$
$$f = 6r_3 \tag{3.56}$$

Again, the same arguments apply as for z odd and the set $\overline{4}\ \overline{3}\ \overline{4}$.

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