NNTDM 2 (1996) 1, 28-32

NOTE ON SOME CLASSICAL ARITHMETIC FUNCTIONS

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Let $\{\vartheta_t\}_{t=1}^{\infty}$ be an infinite sequence of real numbers, which satisfies the conditions:

•e_1) For every $t \in \mathcal{N}$ we have $\vartheta_t \in (1, \frac{1+\sqrt{5}}{2})$, where $\mathcal{N}$ denotes the set of natural numbers, i.e. $\mathcal{N} = \{1, 2, 3, \ldots\}$;

•e_2) For every $t \in \mathcal{N}$ it is fulfiled

$$\frac{1}{\vartheta_{t+1} - 1} - \frac{1}{\vartheta_t - 1} \geq 1$$

(in particular from e_2) immediatly follows that for every $t \in \mathcal{N}$, $\vartheta_t > \vartheta_{t+1}$).

•e_3) The sequence $\{a_n\}_{n=1}^{\infty}$ converges to $+\infty$, where

$$a_n = \vartheta_1.\vartheta_2 \ldots \vartheta_n, \qquad n \in \mathcal{N}. \tag{1}$$

From (1) obviously follow

$$a_n = \vartheta_n.a_{n-1}, \qquad n \in \mathcal{N}, \qquad n \geq 2 \tag{2}$$

and

$$a_{n+1} > a_n, \qquad n \in \mathcal{N}. \tag{3}$$

Definition: Let $a \in \mathcal{N}$ be fixed. For every arithmetic function f we set

$$F_f(a) = \{x \in \mathcal{N} : [f(x)] = a\} \tag{4}$$

(here and further we denote by $[y]$ the integer part of y).

The main result of this paper is :

Theorem 1. Let $\{p_t\}_{t=1}^{\infty}$ be an increasing sequence of primes and $\{\vartheta_t\}_{t=1}^{\infty}$ satisfies the conditions e_1), e_2), and e_3). If a multiplicative function f satisfies the relations

$$f(p_t) = \vartheta_t, \qquad t \in \mathcal{N} \tag{5}$$

then for every $a \in \mathcal{N}$ the set $F_f(a)$ has infinitely many elements x, for which it is fulfiled

$$\mu(x) = 0, \tag{6}$$

where μ is the classical Miobius function.

Let $m \in \mathcal{N}$ with canonical form $m = q_1^{\alpha_1}.q_2^{\alpha_2} \ldots q_k^{\alpha_k}$ (q_ν are different primes, $\alpha_\nu \geq 1$ are natural numbers, and $\nu = 1, 2, \ldots, k$) be arbitrary. We consider the following classical arithmetic functions given bellow:

$$\psi(m) := m(1 + \frac{1}{q_1})(1 + \frac{1}{q_2}) \ldots (1 + \frac{1}{q_k}) \tag{7}$$

$$\varphi(m) := m(1 - \frac{1}{q_1})(1 - \frac{1}{q_2}) \ldots (1 - \frac{1}{q_k}) \tag{8}$$

$$\Phi(m) := m^2(1 - \frac{1}{q_1^2})(1 - \frac{1}{q_2^2})\dots(1 - \frac{1}{q_k^2}) \tag{9}$$

$$\sigma(m) := \frac{q_1^{\alpha_1+1} - 1}{q_1 - 1}\frac{q_2^{\alpha_2+1} - 1}{q_2 - 1}\dots\frac{q_k^{\alpha_k+1} - 1}{q_k - 1} \tag{10}$$

where $\sigma(m)$ is the sum of all divisors of m, $\Phi(m)$ equals to the quantity of all irreductible fractions in the matrix:

$$\begin{pmatrix} \frac{1+i}{m}, & \frac{1+2i}{m}, \dots, & \frac{1+mi}{m} \\ \frac{2+i}{m}, & \frac{2+2i}{m}, \dots, & \frac{2+mi}{m} \\ \vdots & & \\ \frac{m+i}{m}, & \frac{m+2i}{m}, \dots, & \frac{m+mi}{m} \end{pmatrix}, (i = \sqrt{-1}),$$

$\varphi(m)$ is the Euler's totient function, $\psi(m)$ is the function of Dedekind.

In the present paper is shown that in Theorem 1., instead of f, we could put everyone of the following functions:

$\frac{\psi(m)}{\varphi(m)}, \frac{\sigma(m)}{\varphi(m)}, \frac{\sigma^2(m)}{\Phi(m)}, \frac{\psi(m)}{m}, \frac{m}{\varphi(m)}, \frac{\sigma(m)}{m}$, (see Theorem 2.)

In order to prove Theorem 1. we need some assertions.

Lemma 1. If $\{\vartheta_i\}_{i=1}^{\infty}$ satisfies the conditions e_1) and e_2) and $\{a_n\}_{n=1}^{\infty}$ is given by (1), then for every $n \in \mathcal{N}$, $n \geq 2$, the inequality

$$a_{n-1} < \frac{2 - \vartheta_n}{\vartheta_n - 1} \tag{11}$$

holds.

Proof: We shall use an induction by n. From e_1) we obtain

$$\vartheta_1 < \frac{1}{\vartheta_1 - 1}. \tag{12}$$

On the other hand the inequality

$$\frac{1}{\vartheta_1 - 1} \leq \frac{2 - \vartheta_2}{\vartheta_2 - 1} \tag{13}$$

holds, because of e_2). Therefore (12) and (13) imply the inequality

$$\vartheta_1 < \frac{2 - \vartheta_2}{\vartheta_2 - 1}. \tag{14}$$

But (14) coincided with (11) when $n = 1$.

Let (11) be fulfiled for some $n \geq 2$. It remains to prove that (11) holds with $n + 1$ instead of n too.

We multiply the both hands of (11) by ϑ_n and using (2) we obtain:

$$a_n < \vartheta_n.\frac{2 - \vartheta_n}{\vartheta_n - 1}. \tag{15}$$

Since

$$\frac{2-\vartheta_{n+1}}{\vartheta_{n+1}-1} = \frac{1}{\vartheta_n - 1} - 1,$$

the property e_2) implies the inequality

$$\frac{2-\vartheta_{n+1}}{\vartheta_{n+1}-1} \geq \frac{1}{\vartheta_n - 1}. \tag{16}$$

Since $\vartheta_n < 1$, from the obvious inequality

$$\vartheta_n(2-\vartheta_n) < 1$$

it follows

$$\vartheta_n . \frac{2-\vartheta_n}{\vartheta_n - 1} < \frac{1}{\vartheta_n - 1}. \tag{17}$$

But (16) and (17) yield

$$\vartheta_n . \frac{2-\vartheta_n}{\vartheta_n - 1} < \frac{2-\vartheta_{n+1}}{\vartheta_{n+1}-1}. \tag{18}$$

Hence

$$a_n < \frac{2-\vartheta_{n+1}}{\vartheta_{n+1}-1}, \tag{19}$$

because of (15).

The inequality (19) is just the same as (11) with $n+1$ instead of n.

The lemma is proved.

Lemma 2. Let $a \in \mathcal{N}$ and $a \geq 2$ be fixed. If there exists $n \geq 2$, such that the inequality

$$a - 1 \leq a_{n-1} < a \tag{20}$$

holds, then the inequality

$$a_n \geq a + 1 \tag{21}$$

is impossible.

Proof: Let (20) holds for some $n \geq 2$. Then

$$\vartheta_n . a_{n-1} < \vartheta_n . a$$

hence

$$a_n < \vartheta_n . a, \tag{22}$$

because of (2).

Let us suppose that (21) holds. Then (21) and (22) imply

$$a + 1 < \vartheta_n . a$$

hence

$$a > \frac{1}{\vartheta_n - 1}. \tag{23}$$

But (20) yields

$$1 + a_{n-1} \geq a. \tag{24}$$

Therefore from (23) and (24) we obtain

$$1 + a_{n-1} > \frac{1}{\vartheta_n - 1}.$$

Hence

$$a_{n-1} > \frac{2-\vartheta_n}{\vartheta_n - 1}.$$

But the last inequality contradicts to (11), proven in Lemma 1.

The lemma is proved.

Corollary If a_{n-1} satisfies (20) for some $n \geq 2$ and $a_n \geq a$, then

$$[a_n] = a. \qquad (25)$$

Lemma 3. Let $\{\vartheta_t\}_{t=1}^{\infty}$ be an arbitrary sequence, which satisfies $e_1), e_2)$ and $e_3)$. Then for every $a \in \mathcal{N}$ there exists n such, that (25) holds.

Proof: We shall use an induction by a.

For $a = 1$ we set $n = 1$ and have $[a_1] = [\vartheta_1] = 1$, i.e. the assertion of the lemma is true, when $a = 1$.

Let the lemma is true for some $a - 1$, where $a - 1 \geq 1$, i.e. there exists n such, that the equality

$$[a_{n-1}] = a - 1 \qquad (26)$$

holds.

Let n denotes the greatest number for which (26) is fulfiled. Such n always exists, because of e_3). For this n we obviously have

$$a_n \geq a. \qquad (27)$$

But (26) is equivalent with (20). Therefore the corollary of Lemma 2. is applicable, because of (27). Hence

$$[a_n] = a.$$

The lemma is proved.

Proof of Theorem 1. Let $a \in \mathcal{N}$ be fixed, $j \geq 0$ be integer. Instead of $\{\vartheta_t\}_{t=1}^{\infty}$ we consider the sequence $\{\vartheta_{j+t}\}_{t=1}^{\infty}$.

If we set

$$a_{n,j} = \vartheta_{j+1}.\vartheta_{j+2} \ldots \vartheta_{j+n}, \qquad n \in \mathcal{N}, \qquad (28)$$

then the conditions $e_1), e_2)$ and $e_3)$ are satisfied for the sequence $\{\vartheta_{j+t}\}_{t=1}^{\infty}$, but with ϑ_{j+t} instead of ϑ_t and $a_{n,j}$ instead of a_n.

Therefore the assertion of Lemma 3. remains valid, hence

$$[a_{n,j}] = a \qquad (30)$$

for a suitable n.

Using (28) we rewrite (29) in the form

$$[\vartheta_{j+1}.\vartheta_{j+2} \ldots \vartheta_{j+n}] = a \qquad (30)$$

and set

$$x_j = p_{j+1}.p_{j+2} \ldots p_{j+n}. \qquad (31)$$

Since f is a multiplicative function, we rewrite (30) as

$$[f(x_j)] = a, \qquad (32)$$

because of (5) (with $j + t$ instead of t) and (31).

It is clear that $\mu(x_j) = 0$, because of (31) and $x_j \in F_f(a)$, because of (32).

Now, let j runs the set $\{0, 1, 2, \ldots\}$. For every j we have $x_j \in F_f(a)$. The theorem is proved, since $x_{j_1} \neq x_{j_2}$, when $j_1 \neq j_2$.

We are ready to get a corollary from Theorem 1.

• I. Let $\{p_t\}_{t=1}^{\infty}$ be an increasing sequence of consequtive primes and $p_1 \geq 5$. We set

$$\vartheta_t = \frac{p_t + 1}{p_t - 1}, \qquad t \in \mathcal{N}.$$

Obviously the conditions $e_1), e_2)$ and $e_3)$ are satisfied for the sequence $\{\vartheta_t\}_{t=1}^{\infty}$. If we put $f(m) = \dfrac{\psi(m)}{\varphi(m)}$, $f(m) = \dfrac{\sigma(m)}{\varphi(m)}$ or $f(m) = \dfrac{\sigma^2(m)}{\Phi(m)}$ (see (7) , (8), (9) and (10)), then relations (5) hold.

• II. Let $\{p_t\}_{t=1}^{\infty}$ be an increasing sequence of consequtive primes and $p_1 \geq 2$. We set

$$\vartheta_t = 1 + \frac{1}{p_t}, \qquad t \in \mathcal{N}.$$

Obviously the conditions $e_1), e_2)$ and $e_3)$ are satisfied for the sequence $\{\vartheta_t\}_{t=1}^{\infty}$. If we put $f(m) = \dfrac{\psi(m)}{m}$ or $f(m) = \dfrac{\sigma(m)}{m}$ (see (7) and (10)), then the relations (5) hold.

• III. Let $\{p_t\}_{t=1}^{\infty}$ be an increasing sequence of consequtive primes and $p_1 \geq 3$. We set

$$\vartheta_t = 1 + \frac{1}{p_t - 1}, \qquad t \in \mathcal{N}.$$

Obviously the conditions $e_1), e_2)$ and $e_3)$ are satisfied for the sequence $\{\vartheta_t\}_{t=1}^{\infty}$. If we put $f(m) = \dfrac{m}{\varphi(m)}$, (see (8)), then the relations (5) hold.

So to everyone of the cases I., II. and III. Theorem 1. is applicable and as a result we obtain the following

<u>Theorem 2.</u> Let f be one of the following functions:

$$\frac{\psi(m)}{\varphi(m)}, \frac{\sigma(m)}{\varphi(m)}, \frac{\sigma^2(m)}{\Phi(m)}, \frac{\psi(m)}{m}, \frac{m}{\varphi(m)}, \frac{\sigma(m)}{m}$$

(see (7), (8), (9) and (10)). For every $a \in \mathcal{N}$ the set $F_f(a)$ has infinitely many elements x , for which is fulfiled $\mu(x) = 0$, where μ is the Miobius function.