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# ON TWO ARITHMETIC SETS

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Let the natural number n have the canonical representation $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, where $p_1, \ldots, p_k$ are different prime numbers and $\alpha_1, \ldots, \alpha_k \geq 1$ are natural numbers.

As it is well known, the functions $\psi$, $\varphi$ and $\sigma$ defined for a natural number n by:

$$\psi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}.(p_i + 1),$$

$$\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}.(p_i + 1),$$

$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i + 1} - 1}{p_i - 1}$$

are multiplicative.

Let for the fixed natural number a the two arithmetic sets

$$A_a = \{x / [\frac{\psi(x)}{\varphi(x)}] = a\} \text{ and } B_a = \{x / [\frac{\sigma(x)}{\varphi(x)}] = a\}$$

are defined.

In the paper the following two questions are discussed:

1. Are $A_a \neq \emptyset$ and $B_a \neq \emptyset$ for every natural number a?
2. Is $card(A_a) = card(B_a) = \infty$ for every natural number a, where card(X) is the cardinality of the set X?

Let $\{p_i\}_{i=1}^{\infty}$ be an infinite sequence of prime numbers, satisfying the inequalities: $5 \leq p_1 < p_2 < p_3 < \ldots$

Let the sequence $\{a_i\}_{i=1}^{\infty}$ be defibed by:

$$a_n = \frac{p_1 + 1}{p_1 - 1} . \frac{p_2 + 1}{p_2 - 1} . \ \ldots \ . \frac{p_n + 1}{p_n - 1} \qquad (1)$$

for the natural number $n \geq 1$. Obviously, for $a_n$ it is valid the equality:

$$a_n = \prod_{i=1}^{n} (1 + \frac{2}{p_i - 1}),$$

from where it follows that $\{a_i\}_{i=1}^{\infty}$ is monotone increasing sequence, all multipliers of which are greater than 1 . Therefore, this sequence converges to $\infty$, if $\{p_i\}_{i=1}^{\infty}$ are consequtive prime numbers.

The following recurrent equality holds for $n \geq 2$:

$$a_n = \frac{p_n + 1}{p_n - 1} . a_{n-1} \tag{2}$$

LEMMA 1: For $n \geq 2$ the inequality

$$a_{n-1} < \frac{p_n - 3}{2} \tag{3}$$

is valid.

Proof: We shall use the induction. For $n = 2$ (3) has the form:

$$a_1 = \frac{p_1 + 1}{p_1 - 1} < \frac{p_2 - 3}{2},$$

From $p_2 \geq p_1 + 2$ and from obvious inequality for $p_1 \geq 5$: $p_1^2 - 4.p_1 - 1 > 0$ it follows that

$$\frac{p_1 + 1}{p_1 - 1} < \frac{(2 + p_1) - 3}{2} \leq \frac{p_2 - 3}{2},$$

i.e., $a_1 < \dfrac{p_2 - 3}{2}$ .

Let (3) be valid for some $n \geq 2$. We shall prove (3) for $n + 1$. Let us multiply both sides of (3) with $\dfrac{p_n + 1}{p_n - 1}$ and then use (2). We get:

$$a_n < \frac{p_n - 3}{2} . \frac{p_n + 1}{p_n - 1}. \tag{4}$$

From the obvious inequalities $(p_n - 3).(p_n - 1) < (p_n - 1)^2$ and $p_{n+1} \geq 2 + p_n$ it follows that

$$\frac{p_n - 3}{2} . \frac{p_n + 1}{p_n - 1} < \frac{(2 + p_n) - 3}{2} < \frac{p_{n+1} - 3}{2} .$$

From heve and (4) it follows that $a_n < \dfrac{p_{n+1} - 3}{2}$ with which Lemma 1 is proved.

LEMMA 2. Let $a \geq 2$ is an arbitrary natural number and let for some natural number $n \geq 2$ be valid the inequality:

$$a - 1 \leq a_{n-1} < a. \tag{5}$$

Then the inequality $a_n \geq a + 1$ is not possible.

Proof: From (5) it follows that

$$\frac{p_n + 1}{p_n - 1} . a_{n-1} < a . \frac{p_n + 1}{p_n - 1} . \qquad (6)$$

From (2) and from the last inequality we obtain that

$$a_n < \frac{p_n + 1}{p_n - 1} . a . \qquad (7)$$

Let us assume that (6) is valid. From (6) and (7) follows the inequality $a + 1 < \frac{p_n + 1}{p_n - 1} . a$. Hence

$$a > \frac{p_n + 1}{2} . \qquad (8)$$

On the other hand, from (5) we obtain

$$1 + a_{n-1} \geq a . \qquad (9)$$

From (8) and (9) it follows the inequality $1 + a_{n-1} > \frac{p_n - 1}{2}$, which can be represented in the form $a_{n-1} > \frac{p_n - 3}{2}$, but it is in a contradiction with (3) from Lemma 1. Therefore, our assumption is not valid and hence the Lemma 2 is proved.

COROLLARY 1: If $a_{n-1}$ satisfies (5), then $a_n$ satisfies the same inequality or it is valid that $a \leq a_n \leq a + 1$.

COROLLARY 2: If $a_{n-1}$ satisfies (5), and $a_n$ satisfies the inequality $a \leq a_n$, then it is valid the equality

$$[a_n] = a . \qquad (10)$$

THEOREM 1: Let $a \geq 1$ be an arbitrary natural number, and $\{p_i\}_{i=1}^{\infty}$ is an arbitrary sequence of consequtive prime numbers for which $5 \leq p_1 < p_2 < \ldots$ Then there exists a natural number n, for which $[a_n] = a$.

Proof: We shall use an induction for a.

When $a = 1$, we put $n = 1$ and we have $a_1 = \frac{p_1 + 1}{p_1 - 1}$. Obviously, $[a_1] = 1$.

Let us assume that the theorem is valid for some natural number $a - 1$ and let the natural number $n$ is the greatest one with the property

$$[a_{n-1}] = a - 1 . \qquad (11)$$

Such n exists, because the sequence $\{a_i\}_{i=1}^{\infty}$ is monotone increasing and its limit is $\infty$. Therefore,

$$a_n \geq n . \qquad (12)$$

Obviously, (11) is equivalent to (5) and therefore from (12) and Corollary 2 it follows that (10) holds, with which the Theorem is proved.

Therefore, we get a positive answer to the first of the two questions formulated in the beginning of the paper.

THEOREM 2: For every natural number $a \geq 1$ there are infinitely many natural numbers m, for which $\mu(m) = 0$, where $\mu$ is the Moebius's function, such that

$$[\frac{\psi(n)}{\varphi(n)}] = a \text{ and } [\frac{\sigma(n)}{\varphi(n)}] = a.$$

Proof: Let $\{p_i\}_{i=1}^{\infty}$ be an arbitrary sequence of consequtive prime numbers for which $5 \leq p_1 < p_2 < \dots$ According to Theorem 1 there exists at least one n for which (10) is valid, where $a_n$ is defined by (1). Let us put $m = p_1 . p_2 . \dots . p_n$.

Obviously, $\mu(m) = 0$. It can be seen directly, that

$$\frac{\psi(m)}{\varphi(m)} = \frac{\sigma(m)}{\varphi(m)} = \frac{p_1 + 1}{p_1 - 1} . \frac{p_2 + 1}{p_2 - 1} . \dots . \frac{p_n + 1}{p_n - 1} = a_n,$$

from where it follows that $[\frac{\psi(m)}{\varphi(m)}] = a$ and $[\frac{\sigma(m)}{\varphi(m)}] = a$.

Because there are infinitely many sequences $\{p_i\}_{i=1}^{\infty}$ with consequtive prime numbers for which $5 \leq p_1 < p_2 < \dots$, it follows that there are infinitely many natural numbers m with the above property.

We can note also, that there exists a way for obtaining infinitely many trivial solutions to the two last equalities.

Let a be an arbitrary fixed natural number and let $m = p_1 . p_2 . \dots . p_n$ be a natural number such that $[\frac{\psi(m)}{\varphi(m)}] = a$.

Let for $\alpha = \langle \alpha_1, \alpha_2, \dots, \alpha_n \rangle$ we put $m_\alpha = m(\alpha_1, \alpha_2, \dots, \alpha_n) = \prod_{i=1}^{k} p_i^{\alpha_i}$. Obviously, $m(1, 1, \dots, 1) = m$. Therefore

$$[\frac{\psi(m_\alpha)}{\varphi(m_\alpha)}] = [\frac{\psi(m)}{\varphi(m)}] = [\frac{\sigma(m_\alpha)}{\varphi(m_\alpha)}] = [\frac{\sigma(m)}{\varphi(m)}] = a.$$

Therefore, we have a positive answer to the second of the two questions formulated in the beginning of the paper, too.