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TWO ASYMPTOTIC FORMULAS RELATED TO BI-UNITARY DIVISORS

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1. Introduction

The concept of bi-unitary divisor, introduced by D. Suryanarayana, is a divisor $d > 0$ of the positive integer n such that $d\delta = n$ and $(d,\delta)^{**} = 1$ (where $(a,b)^{**}$ denote the greatest unitary divisor of a and b). D. Suryanarayana [3] established asymptotic formulae for:

$$\sum_{n\le x} \tau^{**}(n)$$

Let $t(n)$ denote the number of ordered pairs of positive integers a and b with least common multiple $[a,b] = n$. It's known that $t(n) = \tau(n^2)$ and several authors established asymptotic formulae for (see [4])

$$\sum_{n\le x} t(n) = \sum_{n\le x} \tau(n^2)$$

We now put the problem that finding asymptotic formulae for:

$$\sum_{n\le x} t^{**}(n)$$

where $t^{**}(n)$ denote the number of positive integers a and b with least common multiple $[a,b] = n$ and $(a,b)^{**} = 1$.

It's easy to observe that:

$$t^{**}(n) = \tau^{**}(n^2)$$

In this paper we establish asymptotic formulae for:

$$\sum_{\substack{n\le x\\(n,m)=1}} t^{**}(n) = \sum_{\substack{n\le x\\(n,m)=1}} \tau^{**}(n^2)$$

and

$$\sum_{n\le x} \tau^{**}(n^4)$$

2. Preliminary lemmas

Lema 2.1.

$$\tau^{**}(n^2) = \sum_{d^2\delta=n} q_2(d)\tau(\delta) \tag{2.1}$$

where $\tau(n)$ the numbers of divisors of n and $q_2(n)$ is the characteristic function of squarefree integers:

$$q_2(n) = \begin{cases} 0 \text{ if } p^2 | n \text{ for some prime } p \\ 1 \text{ if } p^\alpha | n \text{ implies } \alpha = 1 \end{cases}$$

Proof. Because $q_2(n)$ and $\tau(n)$ are multiplicative functions it's sufficient to prove (2.1) for $n = p^\alpha$:

$$\sum_{d^2\delta=p^\alpha} q_2(d)\tau(\delta) = \begin{cases} 1, & \alpha = 0 \\ \tau(p) = 2, & \alpha = 1 \\ \tau(p^\alpha)+\tau(p^{\alpha-1})=2\alpha, & \alpha > 1 \end{cases} \tag{2.2}$$

but:

$$\tau^{**}(n) = \prod_{\substack{p^\alpha \| n \\ \alpha=2k}} \alpha \cdot \prod_{\substack{p^\alpha \| n \\ \alpha=2k+1}} (\alpha+1)$$

which implies:

$$\tau^{**}(n^2) = \prod_{p^\alpha \| n} (2\alpha)$$

and together of (2.2) implies (2.1).

The next lemma, proved in [5, p. 1422, k = 2] involve the following functions:

$\theta(n)$ = the number of square-free divisors of n

$$\psi_2(n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right)$$

$\delta_2(x) = \exp\left\{-A\cdot 2^{-8/5}\cdot \log^{3/5}x(\log\log x)^{-1/5}\right\}$, $A > 0$ is an absolute constant.

Lemma 2.2. For $x \geq 3$, we have:

$$Q_2(x,n) = \sum_{\substack{r \leq x \\ (r,n)=1}} q_2(r) = \frac{6xn}{\pi^2 \psi_2(n)} + O(\theta(n) x^{1/2} \delta_2(x)) \tag{2.3}$$

uniformly in x and n.

Lemma 2.3 ([2], Lemma 3). For $x \geq 3$:

$$\sum_{\substack{m \leq r \\ (m,n)=1}} \tau(m) = \left(\frac{\varphi(n)}{n}\right)^2 x\{\log x + 2\gamma - 1 + 2\alpha(n)\} + O\left(\sum_{d|n} \frac{3^{\omega(d)}}{d^\alpha} x^\alpha\right) \tag{2.4}$$

where $\varphi(n)$ the Euler φ function, γ the Euler constant and:

$$\omega(n) = \sum_{p^\alpha \| n} \alpha, \qquad \alpha(n) = \sum_{p|n} \frac{\log p}{p-1}$$

and $\alpha = \frac{35}{108} + \varepsilon$, and $\alpha \geq \frac{1}{4}$ is true.

Lemma 2.4. For $s > 1$:

$$\sum_{\substack{m=1 \\ (m,n)=1}}^{\infty} \frac{q_2(m)}{m^s} = \frac{\xi(s) n^s}{\xi(2s) \psi_s(n)} \tag{2.5}$$

where

$$\psi_{s+1}(n) = n^s \prod_{p|n} \left(1 + \frac{1}{p^s}\right)$$

Proof. Using the Euler product representation for absolutely convergent series of multiplicative terms [1, Theorem 11.6].

Derivative both sides of (2.5) we obtain:

Lemma 2.5. For $s > 1$

$$\sum_{\substack{m=1 \\ (m,n)=1}}^{\infty} \frac{q_2(m) \log m}{m^s} = \frac{\xi(s)}{\xi(2s)} \cdot \frac{n^s}{\psi_{s+1}(n)} \{a(s,n) - a(s)\} \tag{2.6}$$

where:

$$a(s) = \sum_{p} \frac{\log p}{p^s + 1}, \qquad a(s,n) = \sum_{p|n} \frac{\log p}{p^s + 1}$$

Using Abel summation formula and (2.3) we obtain:

Lemma 2.6. For $x \geq 3$:

$$\sum_{\substack{m>x \\ (m,n)=1}} \frac{q_2(m)}{m^s} = O(\theta(n)x^{1-s}\delta(x)) \tag{2.7}$$

Lemma 2.7. For $x \geq 3$:

$$\sum_{\substack{m>x \\ (m,n)=1}} \frac{q_2(m)\log m}{m^s} = O(\theta(n)x^{1-s}\log x\cdot\delta(x)) \tag{2.8}$$

Lemma 2.8.

$$\tau^{**}(n^4) = \sum_{d^2\delta=n} \mu(d)\tau^2(\delta) \tag{2.9}$$

where $\mu(n)$ the Möbius μ function.

Proof. Because $\mu(n)$ and $\tau(n)$ are multiplicative functions of n, the sum in the lemma is multiplicative, so to complete the proof we simply note that where $n = p^{\alpha}$ the sum has the values indicated in (2.9).

The next for lemmas, proved in [6] involve the following functions:

$\delta(x) = \exp\left\{-A\cdot\log^{3/5}x(\log\log x)^{-1/5}\right\}$, $A > 0$

$\omega(x) = \exp\left\{-A\cdot\log x(\log\log x)^{-1}\right\}$, $A > 0$

$\eta^{(0)}(s) = \eta(s) = \dfrac{1}{\xi(s)}$

$\eta^{(r)}(s) = (\eta^{(r-1)}(s))'$

c_3, c_2, c_1, c_o denote the absolute constants $\left[\text{for example } c_3 = \dfrac{1}{6\xi(2)}\right]$.

Lemma 2.9. For $x \geq 3$

$$\sum_{m\leq x} \tau^2(m) = c_3x\cdot\log^3x+c_2x\cdot\log^2x+c_1x\cdot\log x+c_ox+O(x^{1/2}\delta(x)) \tag{2.10}$$

Lema 2.10. If the Riemann hypothesis is true, then for $x \geq 3$ we have:

$$\sum_{m\leq x} \tau^2(m) = c_3x\cdot\log^3x+c_2x\cdot\log^2x+c_1x\cdot\log x+c_ox+O\left[x^{\frac{2-\alpha}{5-4\alpha}}\,\omega(x)\right] \tag{2.11}$$

Lema 2.11. For $s > 1$, $x \geq 3$:

$$\sum_{m\leq x} \frac{\mu(m)\log^r(m)}{m^s} = (-1)^r \eta^{(r)}(s) + O\left(\frac{\delta(x)\log^r x}{x^{s-1}}\right) \tag{2.12}$$

Lemma 2.12. If the Riemann hypothesis is true, then for $x\geq 3$, $s > 1$ we have:

$$\sum_{m\leq x} \frac{\mu(m)\log^r(m)}{m^s} = (-1)^r \eta^{(r)}(s) + O(x^{1/2-s}\omega(x)\log^r x) \tag{2.13}$$

3. Main results

Theorem 3.1. For $x \geq 3$, $n \geq 1$, $\varepsilon > 0$

$$\sum_{\substack{m\leq x\\(m,n)=1}} \tau^{**}(m^2) = \frac{(\varphi(n))^2}{\psi_2(n)}\cdot\frac{\xi(2)}{\xi(4)}x\{\log x+2\gamma-1+2\alpha(n)+2a(2)-2a(2,n)\}+ \tag{3.1}$$

$$+ O\left(\sum_{d|n} \frac{3^{\omega(d)}}{d^{\alpha}} x^{1/2}\right)$$

uniformly in x and n.

Proof. By (2.1) and (2.4) we have:

$$\sum_{\substack{m\leq x\\(m,n)=1}} \tau^{**}(m^2) = \sum_{\substack{d^2\delta\leq x\\(d\delta,n)=1}} \mu^2(d)\tau(\delta) = \sum_{\substack{d\leq x^{1/2}\\(d,n)=1}} \mu^2(d) \sum_{\substack{\delta\leq \frac{x}{d^2}\\(\delta,n)=1}} \tau(\delta) =$$

$$= \left(\frac{\varphi(n)}{n}\right)^2 x\{\log x +2\gamma-1+2\alpha(n)\} \sum_{\substack{d\leq x^{1/2}\\(d,n)=1}} \frac{\mu^2(d)}{d^2} -$$

$$- 2\left(\frac{\varphi(n)}{n}\right)^2\cdot x\cdot \sum_{\substack{d\leq x^{1/2}\\(d,n)=1}} \frac{\mu^2(d)\log d}{d^2} +$$

$$+ O\left(\sum_{d|n} \frac{3^{\omega(d)}}{d^{\alpha}} \cdot x^{\alpha} \cdot \sum_{\substack{d \le x^{1/2} \\ (d,n)=1}} \frac{\mu^2(d)}{d^{2\alpha}} \right)$$

Now we use the:

$$\sum_{d\le x} \frac{1}{d^s} = O(x^{1-s}) \qquad (0 < s < 1)$$

and (2.5), (2.7), (2.6) and (2.8) we obtain (3.1).

When n=1, Theorem 3.1 gives the following corollary for $x \ge 3$:

$$\sum_{m\le x} \tau^{**}(m^2) = x\,\frac{\xi(2)}{\xi(4)}\,\{\log x + 2\gamma - 1 + 2a(2)\} + O(x^{1/2})$$

Theorem 3.2. For $x \ge 3$:

$$(3.2) \qquad \sum_{m\le x} \tau^{**}(m^4) = A_3 x\log^3 x + A_2 x\log^2 x + A_1 x\log x + A_o x + O(x^{1/2}\delta(x))$$

where:

$$A_3 = C_3\eta(2), \qquad A_2 = C_2\eta(2) + 6C_3\eta'(2),$$

$$A_1 = C_1\eta(2) + 4C_2\eta'(2) + 12C_3\eta''(2)$$

$$A_o = C_o\eta(2) + 2C_1\eta'(2) + 4C_2\eta''(2) + 8C_3\eta'''(2)$$

Proof. Using the method described in [4] we have:

$$\sum_{m\le x} \tau^{**}(m^4) = \sum_{\substack{d^2\delta\le x \\ d\le \rho z}} \mu(d)\tau^2(\delta) + \sum_{\substack{d^2\delta\le x \\ d\le \rho^{-2}}} \mu(d)\tau^2(\delta) - \sum_{\substack{\delta\le \rho^{-2} \\ d\le \rho z}} \mu(d)\tau^2(\delta) =$$

$$= S_1 + S_2 - S_3, \qquad z = x^{1/2}, \qquad 0 < \rho = \rho(x) < 1.$$

For (2.10) and (2.12) we have:

$$(3.3) \quad S_1 = A_3 x\log^3 x + A_2 x\log^2 x + A_1 x\log x + A_o x + O(\rho^{-1} z\delta(\rho z)\log^3 z) + O(\rho^{1-2\alpha} z)$$

$$S_2 = \sum_{\substack{d^2\delta\le x \\ \delta\le \rho^{-2}}} \mu(d)\tau^2(\delta) = \sum_{n\le \rho^{-2}} \tau^2(n) M\left(\left[\frac{x}{n}\right]^{1/2}\right) =$$

$$= O\left\{x^{1/2} \sum_{n\le\rho^{-2}} \tau^2(n)n^{-1/2}\delta\left[\left[\frac{x}{n}\right]^{1/2}\right]\right\}$$

where $M(x) = \sum_{n\le x} \mu(n) = O(x\delta(x))$ (see [7]).

By (2.10) and Abel summation formula we have:

$$\sum_{n\le\rho^{-2}} \tau^2(n)n^{-1/2} = O(\rho^{-1}\log^3\rho^{-2})$$

for which:

(3.4) $$S_2 = O\left\{\rho^{-1}z\delta(\rho z)\log^3 \frac{1}{\rho}\right\}$$

With same procedure:

(3.5) $$S_3 = O\left\{\rho^{-1}z\delta(\rho z)\log^3 \frac{1}{\rho}\right\}$$

Let $\rho = \rho(x) = \left\{\delta\left\{x^{1/4}\right\}\right\}^{1/2}$.

By (3.3), (3.4) and (3.5) we obtain (3.2).

Theorem 3.3. If the Riemann hypothesis is true, then for $x\ge 3$ we have:

$$\sum_{m\le x} \tau^{**}(m^4) = A_3x\log^3x+A_2x\log^2x+A_1x\log x+A_ox+O\left\{x^{\frac{2-\alpha}{5-4\alpha}} \omega(x)\right\}$$

Following the same procedure adopted in Theorem 3.2, and making use of (2.11), (2.13).

4. Conjectures

Our theorems does suggest the following conjectures:

$$\sum_{m\le x} \tau^{**}(m^k) = x \sum_{i=0}^{k-1} A_i(\log x)^i + O(x^{1/2}\delta(x))$$

uniformly in x and k, where A_i, $i\in\{0,\dots,k-1\}$ absolute constants.

If the Riemann hypothesis is true, the conjectured formulas are:

$$\sum_{m\le x} \tau^{**}(m^k) = x \sum_{i=0}^{k-1} A_i (\log x)^i + O\left(x^{\frac{2-\alpha}{5-4\alpha}} \omega(x)\right)$$

uniformly in x and k.

We remark that k = 4 we prove the conjectures.

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