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REMARK ON φ AND ψ FUNCTIONS

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This paper is a continuation of [1,2].

We shall use the following notations (see, e.g., [1]):

For every natural number $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ (where $k, \alpha_1, \alpha_2, \ldots, \alpha_k \geq 1$ are natural numbers, $p_1, p_2, \ldots, p_k$ are different prime numbers):

$$\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}.(p_i - 1);$$

$$\psi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}.(p_i + 1);$$

$$\sigma(n) = \prod_{i=1}^{k} (p_i^{\alpha_i + 1} - 1)/(p_i - 1);$$

$$mult(n) = \prod_{i=1}^{k} p_i;$$

$$sum(n) \equiv sum_1(n) = \sum_{i=1}^{k} p_i;$$

$$cas(n) = k;$$

First, we must agree with the critical note by Prof. J. Sàndor (see [3]) about a mistake from [1], where for every natural number n the inequality (Theorem 2):

$$\psi(n) + \varphi(n) \geq 2.n + 2.sum(n).(cas(n) - 1) \qquad (1)$$

is introduced. Prof. J. Šandor shows that it is not always valid. Really, in the proof of Theorem 2 [1] there is a mistake which cannot be corrected without a change in the form of the inequality. In [3] Prof. Sandor proves the following modification of (1) for every natural number n:

$$\psi(n) + \varphi(n) \geq 2.n + 2.(cas(n) - 1)$$

and in [2] the author introduces another:

THEOREM 2 [2]: For every natural number n, for which $cas(n) = 1$; or $cas(n) = 2$ and $\min(\alpha_1, \alpha_2) \geq 2$; or $cas(n) \geq 3$:

$$\psi(n) + \varphi(n) \geq 2.n + sum(n).(cas(n) - 1).$$

From this assertion it follows

COROLLARY: For every natural number n, for which $cas(n) = 1$; or $cas(n) = 2$ and $\min(\alpha_1, \alpha_2) \geq 2$; or $cas(n) \geq 3$:

$$\sigma(n) + \varphi(n) \geq 2.n + sum(n).(cas(n) - 1)$$

Second, we shall prove two inequalities, related to $\frac{\varphi(n)}{\psi(n)}$. Let in the canonical representation of n: $p_1 < p_2 < \ldots < p_{cas(n)}$.

Let P_s is the product of the first $s + 1$ prime numbers ($\pi_0 = 2$, $\pi_1 = 3$, $\pi_2 = 5$, ...), i.e., $P_s = \prod_{i=0}^{s} \pi_i$.

THEOREM: For every natural number n:

$$\text{(a)} \quad \frac{\varphi(n)}{\psi(n)} > \begin{cases} \dfrac{P_{cas(n)-2}}{3.mult(n)}, & \text{if } n \text{ - even number} \\ \dfrac{P_{cas(n)-1}}{2.mult(n)}, & \text{if } n \text{ - odd number} \end{cases}$$

$$\text{(b)} \quad \frac{\varphi(n)}{\psi(n)} < \begin{cases} \dfrac{3.mult(n)}{2.P_{cas(n)}}, & \text{if } n \text{ - even number} \\ \dfrac{6.mult(n)}{P_{cas(n)+1}}, & \text{if } n \text{ - odd number} \end{cases}$$

Proof: (a) $\dfrac{\varphi(n)}{\psi(n)} = \prod_{i=1}^{k} \dfrac{p_i - 1}{p_i + 1}$.

Obviously, for every natural number a: $\dfrac{a-1}{a+1} < \dfrac{a}{a+2}$. Therefore,

- if n is an odd number, then

$$\frac{\varphi(n)}{\psi(n)} > \prod_{i=1}^{k} \frac{p_i - 2}{p_i} \geq \frac{1.3.\ \cdots\ .\pi_{k-1}}{mult(n)} = \frac{2.3.\ \cdots\ .\pi_{k-1}}{2.mult(n)} = \frac{P_{cas(n)-1}}{2.mult(n)},$$

- if n is an even number, then

$$\frac{\varphi(n)}{\psi(n)} > \frac{1}{3}.\prod_{i=2}^{k} \frac{p_i - 2}{p_i} \geq \frac{2}{3}.\frac{1.3.\ \cdots\ .\pi_{k-2}}{mult(n)} = \frac{2.3.\ \cdots\ .\pi_{k-2}}{3.mult(n)}$$

$$= \frac{P_{cas(n)-2}}{3.mult(n)}.$$

(b). If n is an odd number, then (because $p_1 + 2 \geq 5$)

$$\frac{\varphi(n)}{\psi(n)} < \prod_{i=1}^{k} \frac{p_i}{p_i + 2} \leq \frac{mult(n)}{5.7.\ \cdots\ .\pi_{k+1}} = \frac{6.mult(n)}{2.3.5.7.\ \cdots\ .\pi_{k+1}}$$

$$= \frac{6.mult(n)}{P_{cas(n)+1}}$$

If n is an even number, then

$$\frac{\varphi(n)}{\psi(n)} < \frac{1}{2}.\prod_{i=2}^{k} \frac{p_i}{p_i + 2} \leq \frac{1}{4}.\frac{mult(n)}{5.7.\ \cdots\ .\pi_k} = \frac{3}{2}.\frac{mult(n)}{2.3.5.7.\ \cdots\ .\pi_k}$$

$$= \frac{3.mult(n)}{2.P_{cas(n)}}$$

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[1] Atanassov K., Inequalities for φ, ψ and σ functions. Octogon, Vol. 3, No. 2, Oct. 1995, 11-13.

[2] Atanassov K., Inequalities for φ, ψ and σ functions. submitted to Octogon.

[3] Sándor J., On certain inequalities involving Dedekind's arithmetic function. Notes on Number Theory and Discrete Mathematics, Vol. 2, 1996, No. 1, 1-4.