NNTDM 2 (1996) 1, 1-4

ON CERTAIN INEQUALITIES INVOLVING DEDEKIND'S ARITHMETICAL FUNCTION

József Sándor

4160 Forteni Nr. 79, R-Jud. Harghita, ROMANIA

1. Let $1 < n = \prod_{i=1}^{r} p_i^{a_i}$ be the canonical representation of the positive integer n (p_i primes, $a_i \geq 1$, $r = \omega(n)$ = number of distinct prime factors of n). Dedekind's arithmetical function $\psi(n)$ is defined by

$$\psi(n) = \prod_{i=1}^{r} p_i^{a_i - 1} . (p_i + 1) \tag{1}$$

in analogy with the well known representation of Euler's totient

$$\varphi(n) = \prod_{i=1}^{r} p_i^{a_i - 1} . (p_i - 1). \tag{2}$$

For a survey of results involving function ψ, as well as many new theorems we quite the papers [3-7]. See also the book [2].

Let

$$\beta(n) = p_1 + p_2 + \ldots + p_r = \sum_{p/n} p. \tag{3}$$

This function is connected to many other important number-theoretical functions (see e.g. [2], Ch. IV).

Recently, K. Atanassov [1] has proved that

$$\psi(n) - \varphi(n) \geq 2.(\omega(n) - 1).\beta(n) \tag{4}$$

and

$$\psi(n) + \varphi(n) \geq 2.n + 2.(\omega(n) - 1).\beta(n). \tag{5}$$

Clearly, (5) in this form is not correct, since, e.g., for $n = p.q$ (p, q - distinct primes) one has

$$\psi(n) + \varphi(n) = 2.p.q + 2 \geq 2.p.q + 2.(p + q).$$

Here we shall obtain an algebraic proof of (4) and for a corrected form of (5), with certain remarks on possible refinements. The method we will follow is that of papepr [4]. Our remarks will be based on certain earlier resullts (see e.g. [3]), namely

$$\psi(n) - \varphi(n) \geq \frac{n}{\gamma(n)} . 2^{\omega(n)}, \tag{6}$$

where $\gamma(n) = \prod_{i=1}^{r} p_i$ denotes the "core" of n, function first studied by S. Wiegert (see e.g. [2]);

$$\psi(n) \le \begin{cases} 3^{\omega(n)}.\varphi(n), & \text{for } n \text{ even} \\ 2^{\omega(n)}.\varphi(n), & \text{for } n \text{ odd} \end{cases} \tag{7}$$

$$2^{\omega(n)} \le \frac{\psi(n).d(n)}{\sigma(n)} \le 2^{\Omega(n)} \tag{8}$$

where $d(n) = \prod_{i=1}^{r} (a_i + 1)$ denotes the number of divisors of n,

while $\sigma(n) = \prod_{i=1}^{r} (p_i^{a_i+1} - 1)/(p_i - 1)$ is the sum of divisors of n;

and $\Omega(n) = \sum_{i=1}^{r} a_i$ denotes the total number of prime factors of n.

2. For a new proof of (4), we state the algebraic inequality:

$$\prod_{i=1}^{r} (x_i + 1) - \prod_{i=1}^{r} (x_i - 1) \ge 2.(r - 1).\sum_{i=1}^{r} x_i, \tag{9}$$

for $r \ge 2$, $x_i \ge 2$, which presents an independent interest, too. For $r = 2$ one has equality, and generally this relation can be proved e.g. by induction with respect to r (which we omit here).

Now, by (1), (2) and (9), one can write (when $r \ge 2$):

$$\psi(n) - \varphi(n) = \prod_{i=1}^{r} p_i^{a_i-1}.\left(\prod_{i=1}^{r} (p_i + 1) - \prod_{i=1}^{r} (p_i - 1)\right)$$

$$\ge \prod_{i=1}^{r} p_i^{a_i-1}.2.(r - 1).\beta(n) \ge 2.(\omega(n) - 1).\beta(n)$$

since $a_i \ge 1$. For $r = 1$, relation (4) is trivial.

REMARKS: 1) When all a_i satisfy $a_i \ge 2$ $(1 \le i \le v)$, we get the stronger inequality

$$\psi(n) - \varphi(n) \ge 2.\Gamma(n).(\omega(n) - 1).\beta(n) \tag{10}$$

2) A counter part of (4) is contained in the following:

$$\psi(n) - \varphi(n) \le 2^{\omega(n)}.(n - \varphi(n)) \tag{11}$$

(see [3], p. 10).

3. A corrected form of inequality (5) will be

$$\psi(n) + \varphi(n) \ge 2.n + 2.(\omega(n) - 1). \tag{5'}$$

In order to prove this relation, we state the algebraic inequality

$$\prod_{i=1}^{r} (x_i + 1) + \prod_{i=1}^{r} (x_i - 1) \ge 2.\prod_{i=1}^{r} x_i + 2.(r - 1), \tag{12}$$

for $r \geq 1$, $x_i \geq 2$. Once again, we omit here the simple inductive proof of (12). Now, by (1), (2), and (12), applied for $x_i = p_i \geq 2$ we can write

$$\begin{aligned}
\psi(n) + \varphi(n) &= \prod_{i=1}^{r} p_i^{a_i - 1} . \left(\prod_{i=1}^{r} (p_i + 1) + \prod_{i=1}^{r} (p_i - 1) \right) \\
&\geq \prod_{i=1}^{r} p_i^{a_i - 1} . \left(2 . \prod_{i=1}^{r} p_i + 2.(r - 1) \right) \\
&= 2.n + 2.(r - 1) . \prod_{i=1}^{r} p_i^{a_i - 1} \\
&\geq 2.n + 2.(r - 1)
\end{aligned}$$

by $a_i \geq 1$.

REMARK: When $a_i \geq 2$ for all i $(1 \leq i \leq v)$, is satisfied, then the above procedure yields the inequality

$$\psi(n) + \varphi(n) \geq 2.n + 2.\Gamma(n).(\omega(n) - 1) \tag{13}$$

which is stronger, in this case, than (12).

4. We now indicate certain connections with relations (6) - (8). Inequality (6) is stronger than (4) in the case of $a_i \geq 2$ for all i $(1 \leq i \leq r)$; since $2^r \geq 2.(r - 1)$ and $\prod_{i=1}^{r} p_i^{a_i} \geq \prod_{i=1}^{r} p_i \geq \sum_{i=1}^{r} p_i = \beta(n)$ for $p_i \geq 2$. The validity of $\prod_{i=1}^{r} p_i \geq \sum_{i=1}^{r} p_i$ can be obtained immediately e.g. by induction.

From (7) we get the equivalent forms

$$\psi(n) + \varphi(n) \leq \begin{cases} \varphi(n).(3^{\omega(n)} + 1), & \text{for } n \text{ even} \\ \varphi(n).(2^{\omega(n)} + 1), & \text{for } n \text{ odd} \end{cases} \tag{14}$$

and

$$\psi(n) - \varphi(n) \leq \begin{cases} \varphi(n).(3^{\omega(n)} - 1), & \text{for } n \text{ even} \\ \varphi(n).(2^{\omega(n)} - 1), & \text{for } n \text{ odd} \end{cases} \tag{15}$$

Finally, from (8) we can deduce the somewhat similar inequalities

$$\psi(n) - \frac{\sigma(n)}{d(n)} \begin{cases} \geq (2^{\omega(n)} - 1).\frac{\sigma(n)}{d(n)}, & \text{for } n \text{ even} \\ \leq (2^{\Omega(n)} - 1).\frac{\sigma(n)}{d(n)}, & \text{for } n \text{ odd} \end{cases} \tag{16}$$

These inequalities can be further werkened if we apply the well known double-idequality $n \leq \frac{\sigma(n)}{d(n)} \leq \frac{n+1}{2}$.

5. Finally, we quote two interesting results on $\frac{\varphi(n)}{\psi(n)}$, obtainet by Wigert [8]. Namely,

$$\lim_{n\to\infty} \inf \frac{\varphi(n)}{\psi(n)}.(\log \log n)^2 = \frac{\pi^2}{6}.e^{-2\Gamma} \tag{17}$$

where Γ is Euler's constant; and the following asymptotic result:

$$\sum_{n\leq x} \frac{\varphi(n)}{\psi(n)} = x.\prod_p \left(1 - \frac{2}{p.(p+1)}\right) + O(x^{1/2}.\log x^{3/2}) \tag{18}$$

where p runs through the set of primes.

REFERENCES:

[1] K. Atanassov, Inequalities for φ, ψ and σ functions. Octogon, Vol. 3, No. 2, Oct. 1995, 11-13.

[2] D. Mitrinović, J. Sándor and B. Crstici, Handbook of Number Theory. Kluver Academic Publishers, Dordrecht, 1995.

[3] J. Sándor, On Dedekind's arithmetical function. Univ. Timisoara, Seminarul de Teoria Structurilor No. 51, 1988, 1-15.

[4] J. Sandor, Note on the functions σ and φ. Bulletin of Number Theory and Rel. Topics, Vol. 12, 1988, 78-80.

[5] J. Sandor, On the composition of some arithmetic function. Studia Univ. Babes-Bolyai Math., Vol. 34, 1989, 7-12.

[6] J. Sandor, An application of the Jensen-Hadamard inequality. Nieuw Arch. Wiskunde, Vol. 8, 1990, 63-66.

[7] J. Sandor and R. Sivaramakrishnan, The many facets of Euler's totirnt: III. An assortment of miscellaneous topics. Nieuw Arch. Wiskunde, Vol. 11, 1993, 97-130.

[8] S. Wigert, Note sur deux dunctions arithmetiques. Prace Math.-Fiz., Vol. 38, 1927, 23-29.