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NOTE ON ONE INEQUALITY

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To Sheila Ann and Mitko

Let P be the set of all permutations of the elements of the set $\{1, 2, \ldots, 3n\}$, where n is a natural number. Let

$$S = pr_{1,2,\ldots,2n} P,$$

i.e., let S be a set of the first 2n-projections of the elements of P. Then it is valid the following

THEOREM: For every real numbers $x_1, x_2, \ldots, x_{3n} > 0$:

$$(1 + \prod_{i=1}^{3.n} x_i). \sum_{(i_1,\ldots,i_{2n})\in S} \frac{1}{(\prod_{j=1}^{n} x_{i_j}).(1 + \prod_{j=n+1}^{2.n} x_{i_j})} \geq C_{3n}^{2n} \qquad (1)$$

Proof: First we must note that the cardinality of set P is (3n)! and the cardinality of set S is C_{3n}^{2n}. Let

$$A \equiv (1 + \prod_{i=1}^{3.n} x_i). \sum_{(i_1,\ldots,i_{2n})\in S} \frac{1}{(\prod_{j=1}^{n} x_{i_j}).(1 + \prod_{j=n+1}^{2.n} x_{i_j})} - C_{3n}^{2n}.$$

Therefore,

$$A = (1 + \prod_{i=1}^{3.n} x_i). \sum_{(i_1,\ldots,i_{2n})\in S} \frac{1}{(\prod_{j=1}^{n} x_{i_j}).(1 + \prod_{j=n+1}^{2.n} x_{i_j})} + C_{3n}^{2n}$$

$$- 2.C_{3n}^{2n}$$

$$= \sum_{(i_1,\ldots,i_{2n})\in S} \left(\frac{1 + \prod_{i=1}^{3.n} x_i}{(\prod_{j=1}^{n} x_{i_j}).(1 + \prod_{j=n+1}^{2.n} x_{i_j})} + \frac{(\prod_{j=1}^{n} x_{i_j}).(1 + \prod_{j=n+1}^{2.n} x_{i_j})}{(\prod_{j=1}^{n} x_{i_j}).(1 + \prod_{j=n+1}^{2.n} x_{i_j})}\right)$$

$$- 2.C_{3n}^{2n}$$

$$= \sum_{(i_1,\dots,i_{2n})\in S} \left(\frac{1 + \prod_{j=1}^{n} x_{i_j}}{(\prod_{j=1}^{n} x_{i_j}).(1 + \prod_{j=n+1}^{2.n} x_{i_j})} + \frac{(\prod_{j=1}^{2.n} x_{i_j}).(1+ \prod_{j=2.n+1}^{3.n} x_{i_j})}{(\prod_{j=1}^{n} x_{i_j}).(1 + \prod_{j=n+1}^{2.n} x_{i_j})} \right)$$

$$- 2.C_{3n}^{2n}$$

$$= \sum_{(i_1,\dots,i_{2n})\in S} \left(\frac{1 + \prod_{j=1}^{n} x_{i_j}}{(\prod_{j=1}^{n} x_{i_j}).(1 + \prod_{j=n+1}^{2.n} x_{i_j})} + \frac{(\prod_{j=n+1}^{2.n} x_{i_j}).(1+ \prod_{j=2.n+1}^{3.n} x_{i_j})}{1 + \prod_{j=n+1}^{2.n} x_{i_j}} \right)$$

$$- 2.C_{3n}^{2n}$$

(now, the sum contains $2.C_{3n}^{2n}$ members; we apply the Cauchy inequality)

$$\geq 2.C_{3n}^{2n} . \sqrt[2C_{3n}^{2n}]{\prod_{(i_1,\dots,i_{2n})\in S} \frac{(1 + \prod_{j=1}^{n} x_{i_j}).(\prod_{j=n+1}^{2.n} x_{i_j}).(1 + \prod_{j=2.n+1}^{3.n} x_{i_j})}{(\prod_{j=1}^{n} x_{i_j}).(1 + \prod_{j=n+1}^{2.n} x_{i_j}).(1 + \prod_{j=n+1}^{2.n} x_{i_j})}}$$

$$- 2.C_{3n}^{2n}.$$

Having in mind that the product of all multipliers under the radical is equal to 1, because every multiplier in the numerator is a multiplier in the denominator for a suitable permutation of the elements of the set $\{1, 2, \dots, 3n\}$, we obtain that

$$A = 2.C_{3n}^{2n} - 2.C_{3n}^{2n} = 0,$$

i,e, $A \geq 0$ and therefore, (1) is valid.

When $n = 3$, (1) has the form

$$(1 + x_1.x_2.x_3).\left(\frac{1}{x_1.(1 + x_2)} + \frac{1}{x_2.(1 + x_3)} + \frac{1}{x_3.(1 + x_1)}\right) \geq 3 \qquad (2)$$

where $x_1, x_2, x_3 > 0$ are real numbers.

Obviously, (1) is a generalization of (2). In some sense, another generalization of (2) is the inequality:

$$(1 + \prod_{i=1}^{n} x_i).\sum_{i=1}^{n} \frac{1}{x_i.(1 + x_{i+1})} \geq n, \qquad (3)$$

where $x_{n+1} = x_1$, but the problem for its validity in general is open.

Below we shall give some comments and examples (everywhere x_1, $x_2, \ldots, x_n > 0$).

From

$$(1 + a.b).\left(\frac{1}{a.(b + 1)} + \frac{1}{b.(a + 1)}\right) = \frac{(1 + a.b).(a + b + 2.a.b)}{a.b.(a + 1).(b + 1)}$$

we can easily see, that for a, b $\in$ (0, 1]:

$$\frac{(1 + a.b).(a + b + 2.a.b)}{a.b.(a + 1).(b + 1)} \geq 2,$$

because

$$(a + b + 2.a.b + a^2.b + a.b^2 + 2.a^2.b^2) -$$
$$(2.a.b + 2.a^2.b + 2.a.b^2 + 2.a^2.b^2)$$
$$= a + b - a^2.b - a.b^2 \geq 0,$$

but similarly we can see that for a, b $\geq$ 1:

$$\frac{(1 + a.b).(a + b + 2.a.b)}{a.b.(a + 1).(b + 1)} \leq 2.$$

Let x_1, $x_2, \ldots, x_n \geq 1$, and $x_{n+1} = x_1$, $x_{n+2} = x_2$, $x_0 = x_n$. If for every j ($1 \leq j \leq n$):

$$\prod_{i=1}^{n} x_i + 1 \geq x_j.(x_{j+1} + 1) \tag{4}$$

then (3) holds. For example, (4) will be valid, if x_1, $x_2, \ldots, x_n \geq 2$, and $n \geq 3$. Analogically, if for every j ($1 \leq j \leq n$) $x_j \leq 1$ and $x_j.(x_{j+1} + 1) \leq 1$, then (3) holds.