NNTDM 1 (1995) 3, 142-145

SOME REMARKS CONCERNING FIXED POINTS IN PARTIALLY ORDERED SETS

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November 28, 1995

1 Introduction

In this paper we present theorems about fixed points for the mappings of a poset into itself.
We begin with some definitions from the theory of partially ordered sets,which will be used throughout the paper.The poset denotes a partially ordered set (i.e. a set with a reflexive,antisymmetric and transitive relation $\leq$), 0 and 1 being the least and greatest elements (if they exists) respectively. Let S be a subset of a poset P.An element x of P is an upper(lower) bound of S if $s\leq x$ ($x\leq s$) for all s in S.The terms the least upper bound and the greatest lower bound will be abbrevieted to sup and inf,respectively. We will say that a poset P is chain complete if every chain of P has a sup in P.A mapping f of a poset P into the poset Q is called antitone (isotone) iff for all $x, y\in P$, $x\leq y$ implies $f(y)\leq f(x)$ ($f(x)\leq f(y)$). If f mapping of a poset P into itself, x fixed point if $f(x) = x$.
A mapping f of a poset P into itself is called relatively isotone if $x, y\in P$, $x\leq y, x\leq f(y), f(x)\leq y$ implies $f(x)\leq f(y)$.
We enumerate some results:

Proposition 1. 1 *(Amann [2]) Let* $(P,\leq)$ *a poset such that every chain has upper bound (resp.,lower bound),and let* $f:P\rightarrow P$ *be a map such that*

$$x\leq f(x)(f(x)\leq x)\forall x\in R$$

Then f *has at least one fixed point.*

Theorem 1. 1 *(Kolodner [5] ,Amann [2]) Let* $(P,\leq)$ *a poset such that every chain has an infimum(resp.supremum) and let* $f:P\rightarrow P$ *be isotone.Suppose that there exists an element* $x_0\in P$ *such that*

$$f(x_0)\leq x_0(x_0\leq f(x_0))$$

Then f *has a fixed point.*

Corollary 1. 1 *(Tarski [7]) Let* L *be a complete lattice and let* $f:L\rightarrow L$ *be isotone.Then* f *posseses a least and a greatest fixed point.*

Theorem 1. 2 *(Klimes [3]) Let* L *be a complete lattice,and* $f:L\rightarrow L$ *be antitone mapping. Then there exists a fixed point of* f^2*.*

Same theorems established A.E.Roth [6].

Theorem 1. 3 *(Abian and Brown [1]) Let* P *be a chain complete poset and let* f *be an isotone mapping of* P *into itself.Then* f *has a fixed point.*

Theorem 1. 4 *(Klimes [4]) Let* P *be a chain complete poset and* f *be a relatively isotone mapping of* P *into itself.Then* f *has a fixed point.*

We observe that in Proposition 1.1 we have $x\leq f(x)$ every $x\in P$ and in Theorem 1.1 we have $x_0\leq f(x_0)$ for some x_0 and f isotone; if the mapping antitone we can establish a fixed point theorem to f^2.

2 New fixed point results

From this observations we establish some results.We introduce the concept of weak isotone mapping and establish same theorem for Theorem 1.4.

Definition 2. 1 *Let f be a mapping of a poset P into itself.f is called weak isotone if $x, y \in P, f(x) \leq y$ implies $f^2(x) \leq f(y)$ or $x \leq f(y)$ implies $f(x) \leq f^2(y)$*

Proposition 2. 1 *Let $(P, \leq)$ a poset such that every chain has an upper bound ,and let $f : P \rightarrow P$ be a map such that*

$$\forall x \in P, \exists y \in P : x \leq f(y) and f(x) \leq y$$

Then f^2 has a fixed point.

Proof
Zorn's lemma implies the existence of a maximal element m of P. Then exist $y_1 \in P$ such that:

$$m \leq f(y_1) \text{ and } f(m) \leq y_1.$$

but for maximality of m we have $m = f(y_1)$.Then for $y_1 \in P$ exist $y_2 \in P$ such that

$$y_1 \leq f(y_2) \text{ and } f(y_1) \leq y_2$$

But $m = f(y_1) \leq y_2$ implies $y_2 = m$ and we have $y_1 \leq f(m)$ and $f(m) \leq y_1$ which implies $f(m) = y_1$ or $m = f^2(m)$.

Theorem 2. 1 *Let $(P, \leq)$ a poset such that every chain has an infimum and let $f : P \rightarrow P$ be antitone.If exists $x_0, x_1 \in P$ such that*

$$x_0 \leq f(x_1) and f(x_0) \leq x_1$$

then f^2 has a fixed point.

Proof
f antitone,implies f^2 isotone and $x_0 \leq f(x_1)$ implies $f^2(x_1) \leq f(x_0) \leq x_1$. Because every chain has an infimum Theorem 1.1 implies that f^2 has a fixed point.

Remark
If L be a complete lattice the theorem implies Theorem 1.2.

Proposition 2. 2 *Let $(P, \leq)$ a poset such that every chain has an lower bound and $f : P \rightarrow P$ weak isotone suvh that for every $x \in P$ exist $x_1 \in P$ such that:*

$$f(x_1) \leq x and f(x) \leq x_1.$$

Then f^2 has a fixed point.

Proof
By Zorn'S lemma we have a minimal element m of P.Then

$$\exists x_1 \in P : f(x_1) \leq m, f(m) \leq x_1.$$

Minimality of m implies $f(x_1) = m$.But f weak isotone,$f(m) \leq x_1$ implies $f^2(m) \leq f(x_1) = m$ for which $m = f^2(m)$.

Theorem 2. 2 *Let $(P, \leq)$ a poset such that every chain has an infimum and $f : P \rightarrow P$ be weak isotone.Suppose that there exists an element $x_0, x_1 \in P$ such that:*

$$f(x_1) \leq x_0, f(x_0) \leq x_1.$$

Then f^2 has a fixed point.

Proof
Let $X = \{x \in P l \exists x_1 \in P : f(x_1 \leq x, f(x) \leq x_1\}$ and observe that X is not empty $(x_0 \in X)$ and $f(X) \subset X$, because if if $y \in f(X), \exists x \in X : f(x) = y, x \in X$ implies $\exists x_1 \in P : f(x_1) \leq x, f(x) = y \leq x_1$ implies $f^2(x_1) \leq f(x) = y$ and $f(y) \leq f(x_1)$ which implied $y \in X$. But f weak isotone: $f(x) \leq x_1$ implies $f^2(x) \leq f(x_1) \leq x \forall x \in X$. Proposition 1.1 implies the existence of fixed point to f^2.

References

[1] Abian S.,Brown A.B.,*A theorem on partially ordered sets, with applications to fixed point theorems*,Canad.J.Math.13(1961),78-82.

[2] Amann H.,*Order structures and fixed points*,ATTI del Seminario di Analizi Funzionale e Applicazioni 1977,1-51.

[3] Klimes J.,*Fixed edge theorems for complete lattices*, Arch.mat.17(1981),227-234.

[4] Klimes J.,*Fixed point characterizations of completeness on lattices for relatively isotone mappings*,Arch.Mat.20(1984),125-132.

[5] Kolodner I.,*On completeness of partially ordered sets and fixed point theorems for isotone mappings*,Amer.Math.Monthly,75(1968) ,48-49.

[6] Roth A.E.,*A lattice fixed point theorem with constraints*, Bull.Amer.Math.Soc.,81(1975),136-138.

[7] Tarski A.,*A lattice theoretical fixpoint theorem and its applications*,Pacific J.Math.5(1955),285-309.