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WALKING ON A CHESSBOARD

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1. INTRODUCTION

Italian TV serials seem to be a good source of mathematical problems (e.g., see [4], and the comment [5] by the Editor of the Advanced Problem column of the journal where the problem was published). During the TV serial *Buona Domenica* (Happy Sunday) a player is challenged to walk along a predetermined *path* (which is unknown to him) on a 4-by-10 chessboard. A red light warns him about any off-path step.

While watching this serial, the second author's wife asked the following question.

Question 1. What is the probability q that the red light never glows? In other words, what is the probability that the player travels the right path at his first attempt?

The answer could not be given immediately. In fact, posing the question in a more general form gave rise to this note and the research works [1] and [2]. To our great surprise, the ubiquitous Fibonacci sequence $\{F_j\}$ is involved in the answer to Question 1. Namely, we get $q = 1/(2F_{21})$. As we shall see, letting n be the length of the chessboard ($n = 10$ in the original problem), we get $q = 1/(2F_{2n+1})$. Furthermore, if we let $m = 3$ be the width of the chessboard ($m = 4$ in the original problem), we encounter the Pell sequence $\{P_j\}$. A supposedly new 4-by-4 Fibonacci matrix and some trigonometrical identities emerge from our study as by-product results.

2. THE PROBLEM

Let **A** be an m-by-n matrix (chessboard) with entries $a_{h,k}$ $(1 \le h \le m\,;\ 1 \le k \le n)$. Let a *path* in **A** be defined as an ordered collection of points (entries) $\{v_1, v_2, \dots, v_k, \dots, v_n\}$ subject to the following constraints:

(i) $v_1 = a_{h,1}$ $(1 \le h \le m)$; (2.1)

(ii) For $2 \le k \le n$, the successor v_k of $v_{k-1} = a_{h,k-1}$ is

$$v_k = \begin{cases} a_{1,k} \text{ or } a_{2,k} & (h = 1) \\ a_{m,k} \text{ or } a_{m-1,k} & (h = m) \\ a_{h-1,k} \text{ or } a_{h,k} \text{ or } a_{h+1,k} & (2 \le h \le m-1). \end{cases} \qquad (2.2)$$

To clarify what said above, a path in a 5-by-11 chessboard is shown in Figure 1.

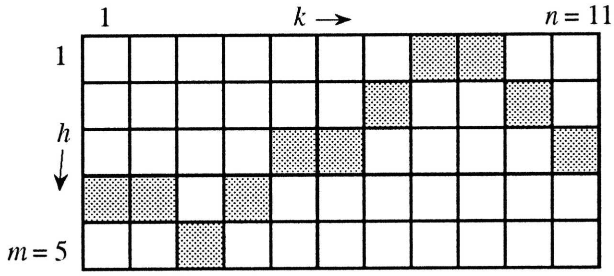

Figure 1. A path in a 5-by-11 chessboard

Question 2. What is the probability that a player travels a predetermined path in **A** at his first attempt?

This probability is clearly given by the reciprocal of the number $p_n(m)$ of all *distinct* paths in **A**. Two paths are said to be distinct if they differ by at least one point v_k.

In the next section an algorithm is given for finding $p_n(m)$.

3. HOW TO FIND $p_n(m)$

It is evident that

$$p_n(m) = \sum_{h=1}^{m} s_{h,n}(m) , \tag{3.1}$$

where $s_{h,n}(m)$ (or simply $s_{h,n}$, if no misunderstanding can arise) is the number of all distinct paths whose terminal point is $a_{h,n}$. From (2.2), we get the relations

$$s_{h,1} = 1 \quad (1 \leq h \leq m) \tag{3.2}$$

and, for $n \geq 2$,

$$s_{h,n} = \begin{cases} s_{1,n-1} + s_{2,n-1} & (h = 1) \\ s_{h-1,n-1} + s_{h,n-1} + s_{h+1,n-1} & (2 \leq h \leq m-1). \\ s_{m-1,n-1} + s_{m,n-1} & (h = m) . \end{cases} \tag{3.3}$$

Now define the vector

$$\mathbf{s}_n = \left[s_{1,n}, s_{2,n}, \ldots, s_{m,n}\right]^{\mathrm{T}} , \tag{3.4}$$

where the superscript $^{\mathrm{T}}$ denotes transposition, and the m-by-m tridiagonal matrix

$$\mathbf{R}_m = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix} . \tag{3.5}$$

From (3.3), (3.4) and (3.5) we can write the matrix difference equation

$$\mathbf{s}_n = \mathbf{R}_m \mathbf{s}_{n-1} \tag{3.6}$$

with initial condition

$$\mathbf{s}_1 = [1, 1, \dots, 1]^{\mathrm{T}} \quad \text{[cf. (3.2)]}, \tag{3.7}$$

whence we get

$$\mathbf{s}_n = \mathbf{R}_m^{n-1} \mathbf{s}_1 . \tag{3.8}$$

Relations (3.8), (3.4) and (3.1) allow us to state the following proposition.

Proposition 1. The number $p_n(m)$ is given by the sum of all the entries of $\mathbf{s}_n$, that is, the sum of all the entries of the $(n-1)$st power of the matrix $\mathbf{R}_m$.

Nowadays, one may have at disposal several software packages which rapidly perform matrix operations. Moreover, since $\mathbf{R}_m$ is an integer matrix, no precision problem can arise, so that rising it to any (reasonably high) power is a rather easy task.

On the other hand, as we shall see in the next section, one may take advantage from the results established in [4] for obtaining a compact form for $p_n(m)$.

4. A COMPACT FORM FOR $p_n(m)$

Properties of a certain class of Toeplitz matrices (which $\mathbf{R}_m$ belongs to) have been thoroughly investigated in [1] (see also [2]). The results pertaining to $\mathbf{R}_m$ that are relevant to this note are reported in Propositions 2 and 3.

Proposition 2. The generic entry $r_{h,k}^{(n)}$ of $\mathbf{R}_m^n$ ($n \geq 0$, an integer) is given by

$$r_{h,k}^{(n)} = \frac{2}{m+1} \sum_{j=1}^{m} \lambda_j^n \sin\frac{jh\pi}{m+1} \sin\frac{jk\pi}{m+1} \quad (1 \leq h, k \leq m), \tag{4.1}$$

where

$$\lambda_j = 1 + 2\cos\frac{j\pi}{m+1} \qquad (j = 1, 2, \ldots, m) \tag{4.2}$$

are the eigenvalues of $\mathbf{R}_m$.

Proposition 3. The sum $\sigma_m^{(n)}$ of all $r_{h,k}^{(n)}$ is given by

$$\sigma_m^{(n)} = \frac{2}{m+1} \sum_{j=1}^{\lfloor (m+1)/2 \rfloor} \lambda_{2j-1}^{n} \cot^2 \frac{(2j-1)\pi}{2m+2}, \tag{4.3}$$

where the symbol $\lfloor \cdot \rfloor$ denotes the greatest integer function.

A compact form for $p_n(m)$ can now be derived from Propositions 1 and 3.

Proposition 4. $p_n(m) = \sigma_m^{(n-1)} = \dfrac{2}{m+1} \displaystyle\sum_{j=1}^{\lfloor (m+1)/2 \rfloor} \lambda_{2j-1}^{n-1} \cot^2 \frac{(2j-1)\pi}{2m+2}$. (4.4)

5. SOME SPECIAL CASES

An explicit form for $p_n(m)$ can be readily obtained for early values of m. The cases $m = 3$ and $m = 4$ are of particular interest to us.

Proposition 5. $p_n(1) = 1 \ \forall n$. (5.1)

Proposition 6. $p_n(2) = 2^n$. (5.2)

Both (5.1) and (5.2) are trivial results. On the other hand, they can be immediately obtained from Proposition 4 and (4.2).

Proposition 7. $p_n(3) = \frac{1}{2} Q_{n+1} = P_n + P_{n+1}$, (5.3)

where Q_n denotes the nth Pell-Lucas number [3].

Proof. Since $\lambda_1 = 1 + \sqrt{2}$, $\lambda_3 = 1 - \sqrt{2}$ [see (4.2)], $\cot(\pi/8) = \sqrt{2} + 1$ and $\cot(3\pi/8) = \sqrt{2} - 1$, from Proposition 4 we get the expression

$$p_n(3) = \frac{1}{2}[(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}]$$

which equals the right-hand side of (5.3) by virtue of the closed-form expressions (Binet forms) for Pell and Pell-Lucas numbers (e.g., see [3]). Q.E.D.

Proposition 8. $p_n(4) = 2F_{2n+1}$. (5.4)

Proof. Since $\lambda_1 = [(1+\sqrt{5})/2]^2 = \alpha^2$, $\lambda_3 = [(1-\sqrt{5})/2]^2 = \beta^2$ [see (4.2)], $\cot^2(\pi/10) = \sqrt{5}\alpha^3$ and $\cot^2(3\pi/10) = \sqrt{5}(-\beta)^3$, from Proposition 4 and the Binet form for Fibonacci numbers, we get

$$p_n(4) = \frac{2}{\sqrt{5}}(\alpha^{2n+1} - \beta^{2n+1}) = 2F_{2n+1} . \quad \text{Q.E.D.}$$

Proposition 9.

$$p_n(5) = \frac{1}{3}[(2+\sqrt{3})^2(1+\sqrt{3})^{n-1} + (2-\sqrt{3})^2(1-\sqrt{3})^{n-1} + 1] . \tag{5.5}$$

Proof. Observe that $\lambda_1 = 1+\sqrt{3}$, $\lambda_3 = 1$, $\lambda_5 = 1-\sqrt{3}$ [see (4.2)], $\cot^2(\pi/12) = (2+\sqrt{3})^2$, $\cot^2(\pi/4) = 1$ and $\cot^2(5\pi/12) = (2-\sqrt{3})^2$, and use Proposition 4. Q.E.D.

Remark. By using standard techniques, it can be proved that $p_n(5)$ obeys the second-order non-homogeneous recurrence relation

$$p_1(5) = 1, p_2(5) = 13 ; p_n(5) = 2p_{n-1}(5) + 2p_{n-2}(5) - 1 \text{ for } n \geq 3 . \tag{5.6}$$

6. RELATED RESULTS

Our study gave rise to the following by-product results.

(i) *Fibonacci matrices* are matrices the entries of the powers of which are related to Fibonacci numbers. It can be proved that $\mathbf{R}_4$ is a Fibonacci matrix (cf. Proposition 8). In [2] we proved that all the entries $r_{h,k}^{(n)}$ of $\mathbf{R}_4^n$ involve Fibonacci numbers. For example, we found the identities

$$r_{11}^{(n)} = r_{44}^{(n)} = (F_{2n-1} + F_{n+1})/2 \tag{6.1}$$

the proofs of which can be readily carried out by using (4.1) and (4.2). We urge the reader to enjoy discovering the "Fibonacci properties" of the matrix $\mathbf{R}_4 - \mathbf{I}_4$ ($\mathbf{I}_4$ being the 4-by-4 identity matrix).

(ii) Letting $n = 0$ and 1 in (4.1) yields the supposedly new trigonometrical identities

$$\sum_{j=1}^{m} \sin\frac{jh\pi}{m+1} \sin\frac{jk\pi}{m+1} = \frac{m+1}{2}\delta_{h,k} , \tag{6.2}$$

($\delta_{h,\,k} = 1$ (0) for $h = (\neq)\ k$ being the Kronecker symbol), and

$$\sum_{j=1}^{m} \cos\frac{j\pi}{m+1} \sin\frac{jh\pi}{m+1} \sin\frac{jk\pi}{m+1} = \frac{m+1}{4}\,\delta_{|h-k|,\,1}\,, \tag{6.3}$$

respectively.

(iii) The numerical evidence emerging from a computer experiment suggests the following conjecture which has been proved by us for $0 \le n \le 3$.

Conjecture 1. The quantity $\sigma_m^{(n)}$ (see Proposition 3) can be expressed as

$$\sigma_m^{(n)} = 3^n m - X_n \quad (m \ge n), \tag{6.4}$$

where $\{X_n\}_0^{\infty} = \{0, 2, 10, 40, 146, 508, 1716, 5682, ...\}$ is a sequence of integers which is *independent* of m.

All our attempts to discover (conjecture) the rule of generation of this sequence were unsuccessful. The interested reader is challenged to achieve this goal, and to prove Conj. 1 for all nonnegative values of n.

As an example of application of Conj. 1 (for the values of n for which it has been proved), we put $n = 0$, 1, 2 and 3 in (6.4), and use (4.3) and some well-known trigonometrical identities to obtain the hopefully new identities

$$\sum_{j=1}^{\lfloor (m+1)/2 \rfloor} \cot^2\frac{(2j-1)\pi}{2(m+1)} = \binom{m+1}{2}, \tag{6.5}$$

$$\sum_{j=1}^{\lfloor (m+1)/2 \rfloor} \cos^2\frac{(2j-1)\pi}{2(m+1)} = \sum_{j=1}^{\lfloor (m+1)/2 \rfloor} \sin^2\frac{(2j-1)\pi}{m+1} = \frac{m+1}{4}, \tag{6.6}$$

and

$$\sum_{j=1}^{\lfloor (m+1)/2 \rfloor} \left[\sin\frac{(2j-1)\pi}{2(m+1)} \sin\frac{(2j-1)\pi}{m+1}\right]^2 = \frac{m+1}{8} \quad (m \ge 3), \tag{6.7}$$

respectively. It has to be noted [see the condition imposed in (6.4)] that the second identity in (6.6) does not hold for $m = 1$.

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