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ON AN ARITHMETIC FUNCTION. PART 2

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Following [1] we shall define the function $l_{a,b,c}$ and here we shall discuss some of its properties.

Let $a, b, c, d \in N$ (everywhere below $N = \{1, 2, 3, \ldots\}$ is the set of the natural numbers) and $(a, b) = (a.b, c) = 1$. Then by $l_{a,b,c}(\alpha)$ we shall denote:

(a) the remain of division of $\mu = \mu_0 . a$ into b, if b is not a divisor of μ;

(b) b, if b is a division of μ,

where (μ_0, γ_0) is an arbitrary solution of

$$a.\mu - b.\gamma = 1. \quad (1)$$

Let everywhere below a, b and c satisfy the above conditions. Therefore $l_{a,b,c}(\alpha)$ is a positive integer and $l_{a,b,c}(\alpha) \in [1, b]$.

In [1] it is proved that:

1. the value of $l_{a,b,c}(\alpha)$ is independent of the choice of the solution (μ_0, γ_0) of (1);
2. function $l_{a,b,c}$ is a bijection on $Y \equiv [1, b]$ over $[1, b]$;
3. for every natural number α, $l_{a,b,c}(\alpha + b) = l_{a,b,c}(\alpha)$;
4. for every two positive integers α_1 and α_2:

$$l_{a,b,c}(\alpha_1 + \alpha_2) = \begin{cases} l_{a,b,c}(\alpha_1) + l_{a,b,c}(\alpha_2), & \text{if } l_{a,b,c}(\alpha_1) + l_{a,b,c}(\alpha_2) \in Y \\ l_{a,b,c}(\alpha_1) + l_{a,b,c}(\alpha_2) - b, & \text{otherwise} \end{cases}$$

5. for every two positive integers α and k:

$$l_{a,b,c}(k.\alpha) = \begin{cases} k.l_{a,b,c}(\alpha), & \text{if } k.l_{a,b,c}(\alpha) \in Y \\ k.l_{a,b,c}(\alpha) - b.s, & \text{otherwise} \end{cases}$$

where $s = [k.l_{a,b,c}(\alpha)/b]$.

Let $m, n \in N$, $m, n \geq 6$ and

$$\frac{1}{2}.(\max(m, n) + 1) = \min(m, n).$$

Let $p \equiv \frac{1}{2}.(\max(m, n) + 1) = \min(m, n)$.

Let $x, y, z \in N$, such that $(x, y) = (x, z) = (y, z) = 1$ and

$$2 \leq x < y < z < y^2 \tag{2}$$

are the solutions of the Diophantine equation

$$x^n + y^m = z^p. \tag{3}$$

Let

$$l_{1,i} = l_{x^{n-1}, y^{4+i}, z^{p-i}}(\alpha) \tag{4}$$

$$l_{2,i} = l_{x^{n-1}, z^{i}, y^{m-4-i}}(\alpha) \tag{5}$$

for $i = 1, 2$.

From the above properties of function $l_{a,b,c}$ it follows that $l_{1,i}$ and $l_{2,i}$ are defined everywhere on N and $l_{1,i} \in [1, y^{4+i}]$, $l_{2,i} \in [1, z^i]$.

Therefore there exist $A_i, B_i \in Z$ (below Z is the set of the integers) such that

$$l_{1,i}.x^{n-1} - A_i.y^{4+i} = \alpha.z^{p-i}$$

$$l_{2,i}.x^{n-1} - B_i.z^{i} = \alpha.y^{m-i-2}$$

for every choice of number $\alpha \in N$ and particularly, for every choice of number $\alpha \in [1, y^{4+i}]$.

We multiply the both sides of the first equation above with z^2 and the both sides of the second equality with y^{4+i} and after this substract the second expression from the first one and as a result we obtain:

$$(l_{1,i}.z^{i} - l_{2,i}.y^{4+i}).x^{n-1} - (A_i - B_i).y^{4+i}.z^{i}$$

$$= \alpha.(z^p - y^m)$$

(from (3))

$= \alpha . x^n$,

from where

$$(l_{1,i} . z^i - l_{2,i} . y^{4+i} - \alpha . x) . x^{n-1} = (A_i - B_i) . y^{4+i} . z^i .$$

From $(x, y.z) = 1$ it follows that

$$\frac{l_{1,i} . z^i - l_{2,i} . y^{4+i} - \alpha . x}{y^{4+i} . z^i} = \frac{A_i - B_i}{x^{n-1}} \in Z \qquad (6)$$

We shall introduce the functions K_i $(i = 1, 2)$ on the set $[1, y^{4+i}] \cap Z$ by

$$K_i(\alpha) = \frac{l_{1,i} . z^i - l_{2,i} . y^{4+i} - \alpha . x}{y^{4+i} . z^i}$$

where $l_{1,i}$ and $l_{2,i}$ are defined by (4) and (5), respectively.

Below we shall study some properties of both new functions, related to function $l_{a,b,c}$.

THEOREM 1: For every $\alpha \in [1, y^{4+i}] \cap Z$, $K_i(\alpha) = 0$ or -1 $(i = 1, 2)$.

Proof: Let $\alpha \in [1, y^{4+i}] \cap Z$. Then

$$K_i(\alpha) = \frac{l_{1,i}}{y^{4+i}} - \frac{l_{2,i}}{z^i} - \frac{\alpha}{y^{4+i}} . \frac{x}{z^i}$$

(from $l_{1,i} \in [1, y^{4+i}] \cap Z$, $l_{2,i} \in [1, z^2] \cap Z$, $\alpha \in [1, y^{4+i}] \cap Z$ and $x < z$ from (2))

$$\leq \frac{y^{4+i}}{y^{4+i}} - \frac{1}{z^i} - \frac{\alpha}{y^{4+i}} . \frac{x}{z^i} < 1,$$

i.e.,

$$K_i(\alpha) \leq 1 \qquad (7)$$

On the other hand,

$$K_i(\alpha) = \frac{l_{1,i}}{y^{4+i}} - \frac{l_{2,i}}{z^i} - \frac{\alpha}{y^{4+i}} . \frac{x}{z^i} \geq \frac{1}{y^{4+i}} - \frac{z^i}{z^i} - \frac{\alpha}{y^{4+i}} . \frac{x}{z^i} > -2,$$

i.e.,

$$K_i(\alpha) > -2 \qquad (8)$$

From (7) and (8) it follows that $K_i(\alpha) \in (-2, 1)$, and from (6) $K_i(\alpha) \in Z$. Therefore $K_i(\alpha) \in \{-1, 0\}$.

THEOREM 2: For every $\alpha \in [1, y^{4+i}] \cap Z$ for which

$$l_{1,i} = l_{x^{n-1}, y^{4+i}, z^{p-i}}(\alpha) \le [\frac{y^{4+i}}{z^i}],$$

holds $K_i(\alpha) = -1$, and

$$l_{2,i} = l_{x^{n-1}, z^i, y^{m-4-i}}(\alpha) \in [z^i - x + 1, z^i] \cap Z.$$

Proof: Let $\alpha \in [1, y^{4+i}] \cap Z$ is such that $l_{1,i} \in [1, [\frac{y^{4+i}}{z^i}]]$. The existence of such α follows from (2). Then

$$K_i(\alpha) = \frac{l_{1,i}.z^i - l_{2,i}.y^{4+i} - \alpha.x}{y^{4+i}.z^i}$$

$$< \frac{\frac{y^{4+i}}{z^i}.z^i - y^{4+i} - \alpha.x}{y^{4+i}.z^i} = \frac{-\alpha.x}{y^{4+i}.z^i} < 0.$$

From $K_i(\alpha) \in \{-1, 0\}$ it follows that $K_i(\alpha) = -1$.

Now, we write the following equivalent equalities:

$$K_i(\alpha) = -1$$

$$\frac{l_{1,i}}{y^{4+i}} - \frac{l_{2,i}}{z^i} - \frac{\alpha}{y^{4+i}}.\frac{x}{z^i} = -1$$

$$l_{2,i} = z^i - \frac{\alpha.x}{y^{4+i}} + \frac{l_{1,i}.z^i}{y^{4+i}}.$$

Therefore,

$$l_{2,i} \ge z^i - x + \frac{z^i}{y^{4+i}}.$$

From (2) it follows that $\frac{z^i}{y^{4+i}} \in (0, 1)$ and therefore,

$$l_{2,i} \geq z^i - x + 1,$$

because $l_{2,i} \in N$. On the other hand, by definition, $l_{2,i} \leq z^i$.

Therefore, $l_{2,i} \in [z^i - x + 1, z^i] \cap Z$.

THEOREM 3: For every $\alpha \in [1, y^{4+i}] \cap Z$, if $K_i(\alpha) = -1$, then

$$l_{2,i} = \in [z^i - x + 1, z^i] \cap Z. \qquad (9)$$

Proof: The existence of values of α which satisfy (9) holds by Theorem 2. Let α_0 is such a number. From the equality

$$\frac{l_{1,i} . z^i - l_{2,i} . y^{4+i} - \alpha_0 . x}{y^{4+i} . z^i} = -1$$

it follows

$$l_{2,i} = z^i - \frac{\alpha_0 . x}{y^{4+i}} + \frac{l_{1,i} . z^i}{y^{4+i}},$$

for where $l_{2,i} \in [z^i - x + 1, z^i] \cap Z$.

THEOREM 4: For every $\alpha \in [1, \frac{y^{4+i}}{x}] \cap Z$, which is not multiple to

z, if $K_i(\alpha) = 0$.

Proof: Let $\alpha_0 \in [1, y^{4+i}] \cap Z$ is not multiple to z. Let us assume that $K_i(\alpha_0) = -1$.

From Theorem 3 it follows that $l_{2,i} \in [z^i - x + 1, z^i] \cap Z$, i. e. $l_{2,i} \neq z^i$, from where

$$K_i(\alpha_0) = -1$$

is equivalent to

$$\frac{l_{1,i}.z^{i} - l_{2,i}.y^{4+i} - \alpha_{0}.x}{y^{4+i}.z^{i}} = -1$$

and also to

$$\alpha_{0} = \frac{l_{1,i}.z^{i} - l_{2,i}.y^{4+i} + y^{4+i}.z^{i}}{x}. \qquad (10)$$

But α_0 is not multiple to z^i, from where it follows that $l_{2,i} \neq z^i$ and hence $l_{2,i} \in [z^i - x + 1, z^i - 1]$, form where (10) obtain the form:

$$\alpha_{0} = \frac{l_{1,i}.z^{i} + (z^{i} - l_{2,i}).y^{4+i}}{x}$$

Therefore,

$$\alpha_{0} > \frac{(z^{i} - l_{2,i}).y^{4+i}}{x}$$

and from $z^i - l_{2,i} \geq 1$ we obtain that

$$\alpha_{0} > \frac{y^{4+i}}{x},$$

which is a contradiction with the condition $\alpha \in [1, \frac{y^{4+i}}{x}] \cap Z$.

Therefore our assumption is not valid, i.e., $K_i(\alpha_0) \neq -1$. Therefore, $K_i(\alpha_0) = 0$.

REFERENCE:

[1] L. Karagyozov, On an arithmetic functions, NNTDM, 1, 1995, 1, 45-47