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# NOTE ON "EXTRAORDINARY PRIMES"

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DEFINITION 1: We shall call an odd prime p an *extraordinary prime of the first kind* iff all terms in the sequence

$$a_k = p + 2^k \tag{1}$$

$(k = 1, 2, 3, \ldots)$ are composite numbers.

DEFINITION 2: We shall call an odd prime q an *extraordinary prime of the second kind* iff all terms in the sequence

$$b_k = |q - 2^k| \tag{2}$$

$(k = 1, 2, 3, \ldots)$ are composite numbers.

DEFINITION 3: We shall call an odd prime r a *super extraordinary prime* iff r is an extraordinary prime of both the first and the second kinds.

OPEN PROBLEM 1: Does there exist at least one super extraordinary prime and if such numbers exist, are they infinitely many.

In this paper it is shown that there are infinitely many extraordinary primes of the first kind (see Theorem 1) and of the second kind, too (see Theorem 2).

At the end of the paper we put two other open problems related to extraordinary primes of the first and of the second kind.

Let

$$F_t := 2^{2^t} + 1 \tag{3}$$

$(t = 0, 1, 2, \ldots)$. For $t \le 4$, $F_t$ are known as Fermat primes. Euler first has shown that $F_5 \equiv 0 \pmod{641}$ (see [1]). Moreover, $F_5 = 641.F$, where 641 and

$$F := 6700417 \tag{4}$$

are primes.

Let

$$G := F_0.F_1.F_2.F_3.F_4.F_5. \tag{5}$$

DEFINITION 4: We introduce the numbers p(t) $(t = 0, 1, 2, \ldots)$ by the equality

$$p(t) := 2.G.t + \frac{102}{641}.G + 1 \tag{6}$$

The explicit form of these numbers is given by

$$p(t) = 36893488147419103230.t + 29353633315419255 31 \qquad (7)$$

DEFINITION 5: We introduce the numbers $q(t)$ $(t = 0, 1, 2, \ldots)$ by the equality

$$q(t) := 2.G.t + \frac{1180}{641}.G - 1 \qquad (8)$$

The explicit form of these numbers is given by

$$q(t) = 36893488147419103230.t + 33958124815877177699 \qquad (9)$$

THEOREM 1: There are infinitely many $t$ for which $p(t)$ is prime. If for any $t$ $p(t)$ is a prime number, then for the same $t$ $p(t)$ is an extraordinary prime of the first kind.

THEOREM 2: There are infinitely many $t$ for which $q(t)$ is prime. If for any $t$ $q(t)$ is a prime number, then for the same $t$ $q(t)$ is an extraordinary prime of the second kind.

Proofs of Theorems 1 and 2 The sequences $p(t)$ and $q(t)$ $(t = 0, 1, 2, \ldots)$, given by (6) and (8), respectively, are arithmetic progressions of a kind $A.t + B$ $(t = 0, 1, 2, \ldots)$. We have $(A, B) = 1$, since $A = 2.G$ and $B = \frac{102}{641}.G + 1$, or $B = \frac{1180}{641}.G - 1$. Therefore, every one of these sequences contains infinitely many primes, because of a well-known Dirichlet's theorem for the prime numbers distribution in an arithmetic progression (see [2]).

Let $t_1$ and $t_2$ be such, that $p = p(t_1)$ and $q = q(t_2)$, given by (6) and (8), are primes. Below we prove that for every $k = 1, 2, 3, \ldots$, $a_k$ from (1) and $b_k$ from (2) are composite numbers.

The set of all natural numbers $N := \{1, 2, 3, \ldots\}$ is a union of the following seven its subsets:

$N_1 = \{1 + 2.m : m \in N_o\}$, $N_2 = \{2 + 4.m : m \in N_o\}$,

$N_3 = \{4 + 8.m : m \in N_o\}$, $N_4 = \{8 + 16.m : m \in N_o\}$,

$N_5 = \{16 + 32.m : m \in N_o\}$, $N_6 = \{32 + 64.m : m \in N_o\}$,

$$N_7 = \{64.m : m \in N\},$$

where $N_o = N \cup \{0\}$; and it is fulfilled:

$N_i \cap N_j = \emptyset$ for $i \neq j$ $(i, j = 1, 2, \ldots, 7)$.

Let $k \in N$ be arbitrary. Then $k \in N_i$ for some $i \in \{1, 2, 3, \ldots, 7\}$. If $i \leq 5$, then one can verify that:

$2^k \equiv -1 \pmod{F_{i-1}}$; $p \equiv 1 \pmod{F_{i-1}}$; $q \equiv -1 \pmod{F_{i-1}}$.

Therefore, in this case, $a_k$ and $b_k$ are composite numbers, because of the congruences:

$$a_k \equiv 0 \pmod{F_{i-1}};\ b_k \equiv 0 \pmod{F_{i-1}}.$$

Let $k \in N_6$. Then it is easy to check that:

$2^k \equiv -1 \pmod{F}$ (see (4)); $p \equiv 1 \pmod{F}$; $q \equiv -1 \pmod{F}$.

Therefore, in this case $a_k$ and $b_k$ are composite numbers too, because of the congruences:

$$a_k \equiv 0 \pmod{F};\ b_k \equiv 0 \pmod{F}.$$

Finally, let $k \in N_7$. In this case we have $2^k \equiv 1 \pmod{641}$.

After some computations we also obtain:

$$p \equiv -1 \pmod{641};\ q \equiv 1 \pmod{641}.$$

Therefore $a_k$ and $b_k$ are composite numbers again, because of of the congruences:

$$a_k \equiv 0 \pmod{641};\ b_k \equiv 0 \pmod{641}.$$

The theorems are proved.

Using (7), it can be shown by a computer that the two first values of t, for which p(t) is prime, are t = 4 and t = 5. Thus, as a corollary from Theorem 1, we obtain that:

p(4) = 15050931592121833845 1 and p(5) = 18740280406863744168 1

are extraordinary primes of a first kind.

In the same way, it can be established that:

q(31) = 117765625738586937782 9, q(61) = 228446090180844247472 9, q(75) = 280096973587230991994 9 are the first three primes from the sequence (9). From Theorem 2 it follows that these numbers are extraordinary primes of the second kind.

OPEN PROBLEM 2: Find the smallest extraordinary prime of the first kind.

OPEN PROBLEM 3: Find the smallest extraordinary prime of the second kind.

REFERENCES:

[1] Edwards H., Fermat's Last Theorem. A Genetic Introduction to Algebraic Number Theory, Springer - Verlag, New York, 1977.

[2] Dirichlet P. G. L., Dedekind R., Vorlesungen uber Zahletheorie, Chelsea, New York, 1968.