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THE FERMAT EQUATION (III)

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ABSTRACT: On the basis of a former paper [1], the author presents a simplified analytical way in order to determine the number of solutions of the Diophantine equation of the title.

1. As was shown in [1], we have that (n is a natural number)

$$F(x) = \frac{[\sqrt[k]{x+0}] + [\sqrt[k]{x-0}]}{2} - \frac{[\sqrt[k]{x-1+0}] + [\sqrt[k]{x-1-0}]}{2} = \begin{cases} 1, & \text{if } n^k < x < n^k + 1 \\ 1/2, & \text{if } x = n^k \text{ or if } x = n^k + 1 \\ 0, & \text{otherwise} \end{cases}, \tag{1}$$

where [u] denotes the greatest integer function.

Hence, the graph of $F(x)$ consists of segments of length 1 and height 1 comprised between the points $x = n^k$ and $x = n^k + 1$; has the value 1/2 at $x = n^k$ or $x = n^k + 1$; and vanishes at all other points.

It follows that if we form:

$$D_k(x_1, x_2) = \frac{1}{2}\{[\sqrt[k]{x_1^k + x_2^k + 0}] + [\sqrt[k]{x_1^k + x_2^k - 0}] + \frac{1}{2}\{[\sqrt[k]{x_1^k + x_2^k - 1 + 0}] + [\sqrt[k]{x_1^k + x_2^k - 1 - 0}] \tag{2}$$

then we have:

$$D_k(x_1, x_2) = \begin{cases} 1, & \text{if } n^k < x_1^k + x_2^k < n^k + 1 \\ 1/2, & \text{if } n^k = x_1^k + x_2^k \text{ or } n^k + 1 = x_1^k + x_2^k \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

from which we derive the twin formulas:

$$\frac{S_k(z_0) + S_k^*(z_0)}{2} = \sum_{2 \le x_1^k + x_2^k \le z_0^k} \sum D_k(x_1, x_2) \tag{4}$$

$$S_k(z_0) + S_k^*(z_0) = \iint_{2 \le x_1^k + x_2^k \le z_0^k} D_k(x_1, x_2) \cdot dx_1 \cdot dx_2 \tag{5}$$

where $S_k(z_0)$ is the number of solutions of the equation $x_1^k + x_2^k = z^k$ for $z \le z_0$; and $S^*_k(z_0)$ is the same thinh for the equation $x_1^k + x_2^k = z^k + 1$ (the solution $x_1^k + x_2^k = z^k$ being counted as different from $x_2^k + x_1^k = z^k$).

2. The calculation of the number of solutions through formula (4) presents unsurmontable difficulties; in change, the way using formula (5) turns out to be eccessible.

The familiar development of the saw-tooth function

$$[x] - x + 1/2 = \sum_{n=1}^{\infty} \frac{\sin(2n\pi x)}{n\pi} \qquad (6)$$

when written as

$$\frac{[x+0] + [x-0]}{2} = x - 1/2 + \sum_{n=1}^{\infty} \frac{\sin(2n\pi x)}{n\pi} \qquad (7)$$

is valid for all real values of x, including the points of discontinuity.

Taking account of what is stated in [2], formula (7) can be also written as

$$\frac{[x+0] + [x-0]}{2} = x - 1/2 + \sum_{n=1}^{N} \frac{\sin(2n\pi x)}{n\pi} + O(\frac{1}{N.x}). \qquad (8)$$

We replace now in (8) x by $\sqrt[k]{x_1^k + x_2^k}$ and $\sqrt[k]{x_1^k + x_2^k} - 1$; put the resulting expressions in (2) and (5), and so we obtain:

$$S_k(z_0) + S^*_k(z_0) = \iint_D \{\sqrt[k]{x_1^k + x_2^k} - \sqrt[k]{x_1^k + x_2^k - 1}$$

$$+ \sum_{n=1}^{N} \frac{1}{n.\pi}.\sin(2n\pi.\sqrt[k]{x_1^k + x_2^k}) - \sum_{n=1}^{N} \frac{1}{n.\pi}.\sin(2n\pi.\sqrt[k]{x_1^k + x_2^k - 1})$$

$$+ \frac{1}{N}.O(\frac{1}{\sqrt[k]{x_1^k + x_2^k}})\}\ dx_1.dx_2. \qquad (9)$$

Here D denotes the region $2 \le x_1^k + x_2^k \le z_0^k$.

Now, according to elementary textbooks, we have that the integral of a finite sum is equal to the sum of the integrals of the individual terms, so that we can write:

$$S_K(z_0) + S^*_K(z_0) = \iint_D (I_1 - I_2).dx_1.dx_2 + \sum_{n=1}^{N} \frac{1}{n.\pi}.\iint_D I_3.dx_1.dx_2 - \qquad (10)$$

$$\sum_{n=1}^{N} \frac{1}{n.\pi}. \iint_D I_4.dx_1.dx_2 + O\{\frac{1}{N}. \iint_D I_5.dx_1.dx_2\}$$

with obvious meaning for I_1, I_2, I_3, I_4 and I_5.

3. All the integrals that appear in (10) are of a type studied by Dirichlet and Liouville. The corresponding formula of evaluation can be consulted in [3]; the proof is given in [4]; the explicit calculation for this case was performed in [1].

Finally, we have the following result:

$$\iint_D (I_1 - I_2).dx_1.dx_2 = c_K.\int_2^{z_0^K} \{t^{3/K-1} - (t - 1)^{1/K}.t^{2/K-1}\}dt, \qquad (11)$$

where $c_K = \frac{\Gamma^2(1 + 1/K)}{\Gamma(2/K)}$;

$$\iint_D I_3.dx_1.dx_2 = K.c_K.\int_{2^{1/K}}^{z_0^K} t.\sin(2n\pi t).dt, \qquad (12)$$

$$\iint_D I_4.dx_1.dx_2 = K.c_K.\int_{2^{1/K}}^{z_0^K} (t^K + 1)^{2/K-1}.t^{K-1}.\sin(2n\pi t).dt, \qquad (13)$$

$$\iint_D \frac{dx_1.dx_2}{\sqrt[K]{x_1^K + x_2^K}} = K.c_K.(z_0 - 2^{1/K}). \qquad (14)$$

Replacing the preceding result in (9) we find the formula:

$$S_K(z_0) + S^*_K(z_0) = c_K.\int_2^{z_0^K} \{t^{3/K-1} - (t - 1)^{1/K}.t^{2/K-1}\}dt$$

$$+ \sum_{n=1}^{N} \frac{K.c_K}{n\pi}.\int_{2^{1/K}}^{z_0^K} \{t - (t^K + 1)^{2/K-1}.t^{K-1}\}.\sin(2n\pi t).dt \qquad (15)$$

$$+ O\{\frac{K.c_K}{N}.(z_0 - 2^{1/K})\}$$

$$= c_K.T_0 + \frac{K.c_K}{\pi}.\sum_{n=1}^{N} \frac{T_n}{n} + O\{\frac{K.c_K}{N}.(z_0 - 2^{1/K})\}$$

say.

4. The binomial series expansion of the integrand in T_0 (for $|t| >$ 1) and subsequent integration (permissible by uniform convergence) gives readily:

$$c_K.T_0 = \frac{c_K}{K}.\int_2^{z_0^K}\{t^{3/K-2} + O(t^{3/K-3})\}dt$$

$$= \begin{cases} \dfrac{c_K}{3-K}.(z_0^{3-K} - 2^{3/K-1}) + O(z_0^{2-K}), & \text{if } K \neq 3 \\ \dfrac{c_3}{3}.(\log z_0^3 - \log 2^3) + O(\dfrac{1}{z_0}), & \text{if } K = 3 \end{cases} \qquad (16)$$

So that we have, for large values of z_0:

$$c_K.T_0 = \begin{cases} c_2.z_0 + O(1), & \text{if } K = 2 \\ c_3.\log z_0 + O(1), & \text{if } K = 3 \\ \dfrac{c_K}{(K-3).2^{3/K-1}} + O\ (1), & \text{if } K \geq 4. \end{cases} \qquad (17)$$

5. As regards the terms T_n, we have that the function

$$f(t) = t - (t^K + 1)^{2/K-1}.t^{K-1} \qquad (18)$$

when expanded by the binomial theorem for $|t| > 1$ can be expressed as:

$$f(t) = -(1 - \frac{2}{K}).\frac{1}{1!.t^K} + (1 - \frac{2}{K}).(2 - \frac{2}{K}).\frac{1}{2!.t^{2K}} - \ldots \qquad (19)$$

from which follows that $f(t)$ and all its derivatives are decreasing functions of t in the interval of integration.

Integration by parts of the T_n gives:

$$T_n = \int_{2^{1/K}}^{z_0} \{t - (t^K + 1)^{2/K-1}.t^{K-1}\}.\sin(2n\pi t)dt$$

$$= \int_{2^{1/K}}^{z_0} f(t).\sin(2n\pi t)dt \qquad (20)$$

$$= -\left| f(t).\frac{\cos(2n\pi t)}{2n\pi}\right|_{2^{1/K}}^{z_0} + \frac{1}{2n\pi}.\int_{2^{1/K}}^{z_0} f'(t).\cos(2n\pi t)dt$$

Now:

$$-\left|f(t).\frac{\cos(2n\pi t)}{2n\pi}\right|_{2^{1/K}}^{z_0} = f(2^{1/K}).\frac{\cos(2n\pi 2^{1/K})}{2n\pi} - f(z_0).\frac{\cos(2n\pi z_0)}{2n\pi} \qquad (21)$$

From (18) we deduce

$$f(2^{1/K}) = 2^{1/K}.\{1 - \frac{2^{1-1/K}}{3^{1-2/K}}\} = O_K(1) \qquad (22)$$

because, as $k \to \infty$, $f(2^{1/k}) \to 1/3$.

From (19) we deduce:

$$f(z_0) = O_k\left(\frac{1}{z_0^k}\right).$$

Hence the expression in (21) has the order of magnitude

$$\frac{1}{n}.O_k(1) \qquad (23)$$

Besides, we have in (20)

$$\left|\int_{2^{1/k}}^{z_0} f'(t).\cos(2n\pi t)dt\right| < \left|\int_{2^{1/k}}^{z_0} f'(t)dt\right| = \left|f(t)\right|_{2^{1/k}}^{z_0} \le f(2^{1/k}) \qquad (24)$$

with $f(2^{1/k})$ gives by (22).

Replacing (21) and (24) in (20) we obtain:

$$T_n = \int_{2^{1/k}}^{z_0} \{t - (t^k + 1)^{2/k-1}.t^{k-1}\}.\sin(2n\pi t)dt = \frac{1}{n}.O_k(1). \qquad (25)$$

6. We return now to (15). On account of the former calculation, we can make there $N \to \infty$, in order to get:

$$S_k(z_0) + S*_k(z_0) = c_k.T_0 + k.c_k.O_k(1).\sum_{n=1}^{\infty} 1/n^2 = c_k.T_0 + d_k.O_k(1)$$

$$= \begin{cases} c_2.z_0 + O(1) + d_2.O(1), & \text{if } k = 2 \\ c_3.\log z_0 + O(1) + d_3.O(1), & \text{if } k = 3 \\ \dfrac{c_k}{(k-3).2^{3/k-1}} + O(1) + d_k.O(1), & \text{if } k \ge 4. \end{cases} \qquad (26)$$

$(d_k = \frac{\pi^2}{6}.k.c_k)$.

In the case of $k = 2$, it is known that both $S_k(z_0)$ and $S*_k(z_0)$ are increasing functions of z_0.

In the case $k = 3$, it is known that $S_k(z_0) = 0$, so we can write

$$S*_k(z_0) = c_3.\log z_0 + O(1)$$

deducing thus that the equation $x_1^3 + x_2^3 = z^3 + 1$ has on infinitude of solutions.

In the case $k \ge 4$, follows from (26) that both equations have a finite quantity of solutions.

But in the case of the Fermat equation, it is obvious that the existence of only a (primitive) solutions, implies the existence of infinitely (imprimitive) many others.

Hence, if we conclude that for $k \geq 4$ it has only a finite quantity of solutions, we must conclude also that there are not solutions at all.

The method was extended in [2] to Euler's equation $x_1^k + \ldots + x_m^k = z^k$ where it is proved the abscense of solutions for $m \leq k-2$. In [6] the method is applies to the equation $x^a + y^b = z^c$ proving that when $1/a + 1/b + 1/c < 1$ there are not solutions.

In [7] is considered the case of the equation $x^4 - h^2.q^4 = z^2$, related to congruent numbers h; and still there is the possibility to apply the method to many other Diophantine equation.

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