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A Note on the Distribution of Vertex Distances in Semiregular Polytopes *

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Abstract

A technique akin to Polya's counting method, is proposed for computing the distribution of the distances between vertices in semiregular spherical polytopes.

1 Introduction

Semiregular spherical polytopes were first introduced by A. Boole Stot [1] as geometric bodies derived from and preserving the symmetry group of regular polytopes in n-dimensional spaces. A complete classification for these objects was obtained by P.H. Schoute [5] along with a description of numerous combinatorial properties. D. Slepian [7] resorted to semiregular polytopes in designing sets of signals, called permutation modulation codes, for sending information over noisy channels.

A permutational semiregular polytope is the convex-hull of a finite set $\mathcal{X} = \{\mathbf{X}_i\}_{i=1}^{M}$ of n-dimensional vectors generated by all the possible permutations of the entries in a given vector

$$\begin{array}{ccccccccccccc} \mathbf{X}_e & = & (a_1, a_1, & \dots & , a_1, & a_2, a_2 & \dots & , a_2, & \dots & , a_r, a_r, & \dots & a_r) \\ & & | \longleftarrow & m_1 & \longrightarrow | & \longleftarrow & m_2 & \longrightarrow | & \dots & | \longleftarrow & m_r & \longrightarrow | \end{array} \tag{1}$$

where

$$\sum_{j=1}^{r} m_j = n \quad \text{with} \quad m_j \geq 1 \quad j = 1, \dots, r \ .$$

Entries in $\mathbf{X}_e$ will be taken in the arithmetical sequence $a_j = a_1 + (j-1)$ to achieve the maximum of the minimum Euclidean distance, [3], with

$$a_1 = -\frac{1}{n} \sum_{j=1}^{r} (j-1) m_j$$

*An earlier version of this paper was presented at Third SIAM Conference on discrete Mathematics, Clemson University, South Carolina, USA, 1986 to yield the zero first-order moment

$$\sum_{j=1}^{r} a_j m_j = 0$$

This last equation shows that every code vector belongs to a hyperplane orthogonal to the all-1's vector, thus the actual dimension of the vector space spanned by $\mathcal{X}$ is $n-1$. Moreover, any vector in $\mathcal{X}$ can be obtained by operating on $\mathbf{X}_e$ with a matrix of the natural representation of $\mathcal{S}_n$, the symmetric group of permutations over n objects. In Slepian's view, $\mathcal{X}$ is an $(n-1)$-dimensional group code [6].
Before stating the problem, let us recall some properties of the symmetric group $\mathcal{S}_n$ over n objects. Throughout the paper, we will use the term *standard partition* to refer to a partition like (1) of the entries in a vector $\mathbf{X}$.

1.1 The symmetric group $\mathcal{S}_n$ and codes

We write $\pi(\mathbf{X}_e)$ or $\mathbf{X}_\pi$ for the vector obtained by re-ordering entries in $\mathbf{X}_e$ according to the permutation

$$\pi = \begin{pmatrix} 1 & 2 & \dots & n \\ \pi(1) & \pi(2) & \dots & \pi(n) \end{pmatrix}$$

The symmetric group $\mathcal{S}_n$ acts transitively on $\mathcal{X}$. Given any pair of vectors $\mathbf{X}_\pi$ and $\mathbf{X}_\eta$, the permutation $\sigma = \eta\pi^{-1}$ sends $\mathbf{X}_\pi$ in $\mathbf{X}_\eta$, that is

$$\sigma(\mathbf{X}_\pi) = \sigma(\pi(\mathbf{X}_e)) = \eta(\mathbf{X}_e) = \mathbf{X}_\eta \ .$$

A pair of code vectors, $\mathbf{X}_\sigma = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$ and $\mathbf{X}_\tau = (x_{\tau(1)}, \dots, x_{\tau(n)})$, has integer Euclidean square-distance

$$|\mathbf{X}_\sigma - \mathbf{X}_\tau|^2 = \sum_{i=1}^{n} [x_{\sigma(i)} - x_{\tau(i)}]^2$$

because $x_{\sigma(i)} - x_{\tau(i)} = a_j - a_\ell = j - \ell$ for some j and ℓ.
The subgroup $\mathcal{H} = \{\sigma : \ \sigma(\mathbf{X}_e) = \mathbf{X}_e \ , \ \sigma \in \mathcal{S}_n\}$ of $\mathcal{S}_n$ leaves $\mathbf{X}_e$ invariant. $\mathcal{H}$ is composed of permutations that exchange equal entries, and its order is

$$|\mathcal{H}| = \prod_{i=1}^{r} m_i!$$

The left transversal $\mathcal{T}$ (i.e. the set of left coset leaders) of $\mathcal{H}$ in $\mathcal{S}_n$, generates $\mathcal{X}$ without repetitions because of the decomposition in cosets

$$\mathcal{S}_n = \bigcup_{\tau \in \mathcal{T}} \tau\mathcal{H} \quad \text{and} \quad \tau\mathcal{H} \cap \sigma\mathcal{H} = \emptyset \quad \text{if } \tau \neq \sigma$$

Therefore a Lagrange theorem, [4, p.33], gives

$$|\mathcal{X}| = |\mathcal{T}| = \frac{|\mathcal{S}_n|}{|\mathcal{H}|} = \frac{n!}{\prod_{i=1}^{r} m_i!}$$

1.2 The problem

Our objective is to count the number of points at a fixed distance from a given point $\mathbf{X}_\sigma$. More formally, the problem is to evaluate the cardinality of the vector set $\mathcal{C}(\kappa)$ at the same square distance κ from $\mathbf{X}_\sigma$. In view of the transitive action of $\mathcal{S}_n$ over $\mathcal{X}$, the distance distribution $|\mathcal{C}(\kappa)|$ does not depend on σ, hence, for counting purposes, we will refer to $\mathbf{X}_e$. Distance κ is an integer not greater than, let us say K. In general, it is not easy to express K explicitly in terms of m_i's, but if the following ordering holds

$$m_1 \leq m_r \leq m_2 \leq m_{r-1} \leq \ldots \leq m_i \leq m_{r-i+1} \leq m_{i+1} \leq \ldots$$

we have

$$K = \sum_{i=1}^{r}[2i-(r+2)][2i-(r+1)]m_i$$

The counting procedure for $|\mathcal{C}(\kappa)|$ will be achieved in successive stages. First, we shall consider a partition, induced by $\mathcal{H}$, of $\mathcal{T}$ into subsets $\mathcal{T}(\tau)$. Secondly, we shall show that each $\mathcal{T}(\tau)$ uniquely identifies a monomial in many variables which are indexed by the cycles of $\mathcal{S}_r$. Then we shall explain how to compute $|\mathcal{T}(\tau)|$ from the monomial. Finally, a generating function will be given for the sum of monomials associated to a fixed distance κ. In order to describe this program in detail, the paper is organized as follows: in Section 2 we collect general properties and results on permutation groups and related codes; Section 3 describes the counting; lastly, in Section 4 we comment on the proposed method and give some illustrative examples.

2 Cycles in $\mathcal{S}_n$

A permutation ν of ℓ symbols v_i, is called a cycle of length ℓ, if each v_i is substituted by the next symbol v_{i+1} and v_ℓ by v_1. It is also denoted

$$\nu = (v_1, v_2, ..., v_\ell) \qquad v_i \in Z_n \tag{2}$$

and in this notation any cyclically shifted writing is actually the same permutation, i.e.

$$\nu = (v_{1+t}, v_{2+t}, ..., v_\ell, v_1, \ldots, v_t) \qquad 1 \leq t < \ell$$

Theorem 1 *Every permutation $\pi \in \mathcal{S}_n$ can be decomposed into disjoint cycles consisting of totally different integers,* [4, p.25]*:*

$$\pi = \nu_1\nu_2...\nu_s \tag{3}$$

Distinct cycles commute.

Now, we consider a many-to-one map f from the set $\mathbb{Z}_n = \{1, 2, \ldots, n\}$ onto the set $\mathbb{Z}_r = \{1, 2, \ldots, r\}$, $r < n$, defined as

$$f(k) = h \qquad \text{if the } k\text{-th entry of } \mathbf{X}_e \text{ is equal to } a_h \ .$$

Using f we define a map F that sends a cycle $\nu \in \mathcal{S}_n$ of length ℓ into an element of $\mathbb{Z}_r^\ell$

$$F(\nu) = (f(v_1), f(v_2), ..., f(v_\ell))$$

With reference to (3), we extend F to every permutation $\pi \in \mathcal{S}_n$, by setting

$$F(\pi) = F(\nu_1)F(\nu_2)...F(\nu_s)$$

Note that the order of cycle images in this decomposition is irrelevant. The motivation behind the definition of F is to give a description of $\mathcal{X}$ which overcomes the multiple correspondence between group elements and code vectors.

2.1 Permutation weight

The distance between vectors in $\mathcal{X}$ can be expressed through the Euclidean weight of a permutation.

Definition 1 *The Euclidean weight of a cycle ν is defined to be*

$$w(\nu) = \sum_{i=1}^{\ell}[f(v_{i+1}) - f(v_i)]^2 \qquad v_{\ell+1} = v_1$$

and, in view of (3), the weight of a permutation τ is defined to be

$$w(\tau) = \sum_{i=1}^{s} w(\nu_i)$$

From the definition of f we have:

$$|\mathbf{X}_{\nu_i} - \mathbf{X}_e|^2 = \sum_{j=1}^{\ell}[x_{v_{j+1}} - x_{v_j}]^2 = \sum_{j=1}^{\ell}\{a_1 + f(v_{j+1}) - [a_1 + f(v_j)]\}^2 = \sum_{j=1}^{\ell}[f(v_{j+1}) - f(v_j)]^2$$

then, the square distance between the image $\mathbf{X}_{\nu_i}$ of a cycle $\nu_i = (v_1, v_2, ..., v_\ell)$ and $\mathbf{X}_e$ is equal to the weight of the cycle. Since different cycles ν_i operate on disjoint sets of entries in $\mathbf{X}_e$, the square distance between $\mathbf{X}_\tau$ and $\mathbf{X}_e$ is the weight $w(\tau)$. Moreover, writing

$$|\mathbf{X}_{\nu_i} - \mathbf{X}_e|^2 = \sum_{j=1}^{\ell}[f(v_{j+1}) - f(v_j)]^2 = 2\sum_{j=1}^{\ell} f(v_j)^2 - 2\sum_{j=1}^{\ell} f(v_{j+1})f(v_j)$$

we see that the square distance is an even integer.

2.2 The transversal $\mathcal{T}$

Peculiar properties of the elements of $\mathcal{T}$ essential to our counting are expressed by the following two theorems.

Theorem 2 *Let $\tau^* \in \mathcal{T}$ be factored into disjoint cycles according to theorem 1:*

$$\tau^* = \tau_1^* \tau_2^* \dots \tau_s^*$$

Each cycle of this decomposition *can be taken not to exchange adjacent equal elements in* $\mathbf{X}_e$. *That is, for* $i = 1, 2, \dots, s$ *each cycle* $(v_{i1},\ v_{i2}, \dots\ v_{ij}, \dots)$ *fulfills the condition*

$$f(v_{ij}) \neq f(v_{ij+1}) \quad \forall j \ . \tag{4}$$

PROOF. We can write a cycle $(v_1, v_2, \dots, u, x, y, v, \dots, v_\ell)$ as a product of the form

$$(v_1, v_2, \dots, u, y, v, \dots, v_\ell)(x, y) \tag{5}$$

where (x, y) is a transposition. If $f(x) = f(y)$, then $(x, y) \in \mathcal{H}$. Iterating a decomposition like (5) as many times as necessary, we obtain $\tau_i^* = \tau_i h_i$, with τ_i satisfying condition (4) and $h_i \in \mathcal{H}$. Since, each τ_i operates on disjoint set of integers, we may reorder the elements to write

$$\tau^* = \tau_1^* \tau_2^* \dots \tau_s^* = \tau_1 h_1 \tau_2 h_2 \dots \tau_s h_s = \tau_1 \tau_2 \dots \tau_s h_1 h_2 \dots h_s = \tau h$$

where τ has the claimed property. □

Theorem 3 *Assume that every coset leader $\tau \in \mathcal{T}$ is chosen to accomplish theorem 2. Let $\nu = (t_1, \dots, t_i, \dots, t_j, \dots, t_\ell)$ $(i < j)$ be a cycle which F-image has $f(t_i) = f(t_j) = u \in \mathbb{Z}_r$. Therefore, there is an equivalent permutation $\tilde{\nu}$ which is a product of separated cycles*

$$\tilde{\nu} = (t_i, \dots, t_{j-1})(t_j, \dots, t_\ell, t_1, \dots, t_{i-1})$$

and such that $\mathbf{X}_\nu = \mathbf{X}_{\tilde{\nu}}$.

PROOF. For theorem 2 we can assume $j \neq i + 1$ and write the cycle as

$$\nu = (t_i, \dots, t_j, \dots, t_\ell, t_1, \dots, t_{i-1})$$

Hence, the F-image has the form

$$F(\nu) = (u, \dots, f(t_{j-1}), u, \dots, f(t_\ell), f(t_1), \dots, f(t_{i-1}))$$

which shows that the same effect is obtained by cyclic shifting two separated sets:

$$(u, \dots, f(t_{j-1})) \quad \text{and} \quad (u, \dots, f(t_\ell), f(t_1), \dots, f(t_{i-1})) \tag{6}$$

Therefore, instead of ν we may consider a permutation $\tilde{\nu}$ factored into cyclic permutations

$$\tilde{\nu} = (t_i, \dots, t_{j-1})(t_j, \dots, t_\ell, t_1, \dots, t_{i-1})$$

that operates over two sets of different integers, but giving $\mathbf{X}_\nu = \mathbf{X}_{\tilde{\nu}}$. □

As a result, every cycle ν_i that appears in some $\tau \in \mathcal{T}$ can have the F-image composed of cycles of $\mathcal{S}_r$. Therefore, we shall assume henceforth that every $\tau \in \mathcal{T}$ has the image

$$F(\tau) = C_{j_1} C_{j_2} \dots C_{j_m} \tag{7}$$

where C_{j_i}s are cycles, not necessarily distinct, of $\mathcal{S}_r$.

2.3 $\mathcal{T}(\tau)$

Let $\mathcal{H}\mathbf{X}_\tau$ denote the set of vectors produced by the action of $\mathcal{H}$ over $\mathbf{X}_\tau$. We define $\mathcal{T}(\tau)$ to be the subset of $\mathcal{T}$ that generates, without repetitions, $\mathcal{H}\mathbf{X}_\tau$. It is worth noting that every permutation in $\mathcal{H}$ exchanges only entries within the same subset of the standard partition. Therefore, knowing the composition of any set of the standard partition of $\mathbf{X}_\tau$ is fundamental for computing the cardinality $|\mathcal{T}(\tau)|$.
Let $n_{kj}(\tau)$ be the number of elements filled in the j-th set ending up in the k-th set by τ, then the composition of the j-th set is completely described by the set of integers

$$\mathbf{n}_j = (n_{1j}(\tau), n_{2j}(\tau), \ldots, n_{j-1j}(\tau), n_{jj}(\tau), n_{j+1j}(\tau), \ldots, n_{rj}(\tau)) \tag{8}$$

where

$$n_{jj}(\tau) = m_j - \sum_{k=1, k\neq j}^{r} n_{kj}(\tau)$$

The full action of τ is described by r vectors $\mathbf{n}_j$ $(j = 1, \ldots, r)$ of dimension r, which we call the *composition set.*

An example. Let $r = 3$, the composition set for the monomial $a(12)a(23)$ is the set of three vectors

$$\begin{cases} \mathbf{n}_1 &= (m_1 - 1, 1, 0) \\ \mathbf{n}_2 &= (1, m_2 - 2, 1) \\ \mathbf{n}_3 &= (0, 1, m_3 - 1) \end{cases}$$

The sets $\mathcal{T}(\tau)$ has the following interesting properties:

Proposition 1 .

1. $\mathcal{T}(\tau) = \mathcal{H}\tau\mathcal{H} \cap \mathcal{T}$
2. $\mathcal{T}(\tau) \cap \mathcal{T}(\sigma) = \emptyset \qquad \forall\, \sigma \notin \mathcal{T}(\tau)$
3. *All permutations in* $\mathcal{T}(\tau)$ *have the same* composition set *as* τ
4. *If* $\tau' \neq \tau \in \mathcal{T}(\tau)$ *then, an* $h \in \mathcal{H}$ *exits such that* $\tau' = h\tau h^{-1}$ *and both* τ *and* τ' *have the same cyclic structure.*

PROOF. 1)From $h(\tau(\mathbf{X}_e) = h\tau h'(\mathbf{X}_e)$ for any $h, h' \in \mathcal{H}$ it follows that the action of $\mathcal{H}\tau\mathcal{H}$ on $\mathbf{X}_e$ is same as the action of $\mathcal{T}(\tau)$. Moreover, each $\tau \in \mathcal{T}$ identifies a single element of $\mathcal{X}$, therefore $\mathcal{H}\tau\mathcal{H} \cap \mathcal{T}$ must generate $\mathcal{H}\mathbf{X}_\tau$ without repetitions.
2) It is a trivial consequence of the definition.
3) Since the permutations in $\mathcal{H}$ exchange only entries within the same subset of the standard partition, then it does not alter the *composition set.*
4) Let be $\tau, \tau' \in \mathcal{T}(\tau)$, assume $\mathbf{X}_{\tau'} \neq \mathbf{X}_\tau$, then an $h \in \mathcal{H}$ exits such that $h(\mathbf{X}_\tau) = \mathbf{X}_{\tau'}$ and

$$h(\mathbf{X}_\tau) = h\tau(\mathbf{X}_e) = h\tau h^{-1}(\mathbf{X}_e) = \tau'(\mathbf{X}_e) \qquad \square$$

Due to this proposition, if we disregard the order of the cycles in (7), all permutations in $\mathcal{T}(\tau)$ have the same F-image as τ. Hence, introducing a set of commuting variables $a(C_i)$ indexed by the cycles of $\mathcal{S}_r$, we can associate a unique monomial to $\mathcal{T}(\tau)$:

$$A(\mathcal{T}(\tau)) = \prod_{i=1}^{\Omega} a(C_i)^{\beta_i} \tag{9}$$

where β_j is the multiplicity, possibly zero, of C_j and Ω is the number of distinct cycles in $\mathcal{S}_r$. The number

$$\Omega = \sum_{j=2}^{r}(j-1)!\binom{r}{j} \tag{10}$$

is obtained, observing that each one of the $\binom{r}{j}$ combinations of j out of r distinct numbers from $\mathbb{Z}_r$ corresponds to $(j-1)!$ cycles because by fixing the first element in a given combination, each permutation of the other elements gives a different cycle.
In conclusion, denoting with $w(\mathcal{T}(\tau))$ the common weight $w(\tau)$ of any element of $\mathcal{T}(\tau)$, we have

$$w(\mathcal{T}(\tau)) = \sum_{j=1}^{\Omega} w(C_j)\beta_j$$

3 Counting

The number of points at distance κ from $\mathbf{X}_e$ is obtained from the decomposition

$$\mathcal{C}(\kappa) = \{\mathbf{X}_\tau : \tau \in \mathcal{T} \text{ and } w(\tau) = \kappa\} = \bigcup_{w(\mathcal{T}(\tau))=\kappa} \{\mathbf{X}_\sigma : \sigma \in \mathcal{T}(\tau)\}$$

implied by proposition 1. Since $|\mathcal{T}(\tau)| = |\{\mathbf{X}_\sigma : \sigma \in \mathcal{T}(\tau)\}|$, we have

$$|\mathcal{C}(\kappa)| = \sum_{w(\mathcal{T}(\tau))=\kappa} |\mathcal{T}(\tau)|$$

The action of $\mathcal{H}$ over $\mathbf{X}_\tau$ produces $|\mathcal{T}(\tau)|$ distinct vectors, therefore using the composition set for τ we get

$$|\mathcal{T}(\tau)| = \prod_{j=1}^{r} \frac{m_j!}{n_{jj}(\tau)! \prod_{k=1,k\neq j}^{r} n_{kj}(\tau)!} \tag{11}$$

with the understanding that $(1/(n_{jj}(\tau))!) = 0$ if $n_{jj}(\tau) = -1$ because, we cannot move more than m_j elements of the j-th set.
The number $n_{kj}(\tau)$ in (8) can be mechanically computed from $A(\mathcal{T}(\tau))$ through the functions $\delta_{kj}(C_i)$, defined as follows:

$$\delta_{kj}(C_i) = \begin{cases} 1 & \text{if } j \text{ is next to } k \text{ in cycle } \; C_i \\ 0 & \text{otherwise} \end{cases} \qquad i = 1, 2, \ldots, \Omega$$

for every $j, k = 1, 2, \ldots r$. Hence, from (7) and (9) we compute the numbers $n_{kj}(\tau)$ as

$$n_{kj}(\tau) = \sum_{i=1}^{\Omega} \beta_i \delta_{kj}(C_i) \qquad j \neq k \tag{12}$$

3.1 Generating function

A generating function $G(x)$ can be produced for polynomials B_κ which are sum of monomials $A(\mathcal{T}(\tau))$ with $w(\mathcal{T}(\tau)) = \kappa$, $\forall \kappa$. Recalling that if $a(C_i)$ is associated to weight $w(C_i)$ then $a(C_i)^\lambda$ is associated to weight $\lambda w(C_i)$, hence the generating function for all monomials associated to a cycle $C_i \in \mathcal{S}_r$ is defined as

$$G_{C_i}(x) = 1 + a(C_i)x^{2w(C_i)} + a(C_i)^2 x^{4w(C_i)} + a(C_i)^3 x^{6w(C_i)} + \ldots = \frac{1}{1 - a(C_i)x^{2w(C_i)}}$$

Moreover the correspondence

$$a(C_i)^{\beta_i} a(C_j)^{\beta_j} \Longrightarrow \beta_i w(C_i) + \beta_j w(C_j)$$

implies that $G(x)$, defined as product of cycle generating functions, is the generating function for the sum of monomials which yield to the same distance κ:

$$G(x) = \prod_{i=1}^{\Omega} G_{C_i}(x) = \prod_{i=1}^{\Omega} \frac{1}{1 - a(C_i)x^{w(C_i)}} = \sum_{\kappa=0}^{\infty} B_\kappa x^\kappa$$

Once B_κ is explicitly obtained, by means of (11) and (12), we compute:

$$|\mathcal{C}(\kappa)| = \sum_{\sum A(\mathcal{T}(\tau)) = B_\kappa} \prod_{j=1}^{r} \frac{m_j!}{n_{jj}(\tau)! \prod_{k=1, k\neq j}^{r} n_{kj}(\tau)!}$$

where the summation is extended to all monomials which sum up to B_κ.

4 Conclusions

In this paper we have described a method for counting the number of vertices at fixed Euclidean distance from a given vertex, in semiregular spherical polytopes. The proposed counting procedure is reminiscent of Polya's counting theory [2] although is far less general because it exploits peculiar properties of the natural permutation representation of $\mathcal{S}_n$. Lastly, even if some results can be obtained by direct counting, the machinery proposed makes all computations straightforward and automatic. In conclusion, we give three illustrative examples.

Example 1 - Let us consider $r = 2$ and assume $m_1 \leq m_2$. Then $\Omega = 1$ and we have only one reduced cycle $C = (1,2)$ with weight $w(C) = 1$. The generating function turns out to be

$$G(x) = \frac{1}{1 - a(12)x^2} = \sum_{k=0}^{\infty} a(12)^k x^{2k}$$

From (12) and (11), we get

$$|\mathcal{C}(2k)| = \frac{m_1!}{k!(m_1 - k)!} \frac{m_2!}{k!(m_2 - k)!} \qquad k = 0, 1, 2, \ldots, m_1$$

and

$$|\mathcal{C}(2k)| = 0 \qquad k > m_1$$

The number of vertices at minimum square-distance 2 from $\mathbf{X}_e$ is $|\mathcal{C}(2)| = m_1 m_2$.

Example 2 - Let us consider $r = 3$ and assume $m_1 \leq m_3 \leq m_2$, then $\Omega = 5$ and the generating function is given by the following product

$$\begin{aligned} G(x) &= \frac{1}{1 - a(12)x^2} \frac{1}{1 - a(23)x^2} \frac{1}{1 - a(31)x^4} \frac{1}{1 - a(123)x^6} \frac{1}{1 - a(132)x^6} \\ &= 1 + [a(12) + a(23)]x^2 + [a(12)^2 + a(23)^2 + a(12)a(23) + a(13)]x^4 + \\ & \quad [a(123) + a(132) + a(12)a(13) + a(13)a(23) + a(12)^3 + a(23)^3]x^6 + \ldots \end{aligned}$$

From (12) and (11), we get for the smallest three distances

$$\begin{aligned} |\mathcal{C}(2)| &= m_1 m_2 + m_2 m_3 \\ |\mathcal{C}(4)| &= m_1 m_3 + m_2(m_2 - 1)[\frac{m_1(m_1 - 1) + m_3(m_3 - 1)}{4} + m_1 m_3] \\ |\mathcal{C}(6)| &= m_2(m_2 - 1)(m_2 - 2) + \frac{m_1(m_1 - 1)(m_1 - 2) + m_3(m_3 - 1)(m_3 - 2)}{6} \\ &+ m_1 m_2 m_3 (m_1 + m_3) \end{aligned}$$

Example 3 - Let us consider the evaluation of the number of vertices at minimum distance. The coefficient of interest is B_2. We have

$$G(x) = 1 + \sum_{i=1}^{r-1} a(i, i+1)x^2 + \sum_{k=2}^{\infty} B_{2k} x^{2k}$$

From (12) and (11), we get

$$|\mathcal{C}(2)| = \sum_{i=1}^{r-1} m_i m_{i+1}$$

References

[1] **A. Boole Stot**, Geometrical deduction of semiregular from regular polytopes, *Ver.der Koninklijke Akad. Wetensch. Amsterdam (eerstie sectie)*, vol.11.1, 1910.

[2] **N.G. De Bruijn**, Polya's Theory of Counting, in *Applied Combinatorial Mathematics*, Ed. E.F. Beckenbach, Wiley, New York, 1964, pp.144-236.

[3] **T. Ericson**, Permutation Codes, *Rapport de Recherche INRIA*, Paris, n.2109, November 1993.

[4] **W. Ledermann**, *Introduction to group theory*, Longman, London, 1973.

[5] **P.H. Schoute**, Analytical treatment of the polytopes regularly derived from regular polytopes, *Ver.der Koninklijke Akad.van Wetenschappen te Amsterdam (eertie sectie)*, vol.11.5, 1913.

[6] **D. Slepian**, Group codes for the Gaussian channel, *BSTJ*, vol.47 (1968), pp.575-602.

[7] **D. Slepian**, Permutation Modulation, *IEEE Proc.*, vol.53, pp.228-236, March 1965.