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SOMETHING NEW ABOUT g(k) IN WARING'S PROBLEM

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As usually in Waring's problem, I denote by $g(k)$ the smallest number of non-negative integer k-th powers sufficient to represent every positive integer.

Let $x \geq 0$ be a real number. Further everywhere $\{x\}$ means $x - [x]$, where $[x]$ denotes the greatest integer, which is not greater than x.

For an integer $k \geq 1$ I define the numbers:

$$E(k) := \left(\frac{3}{2}\right)^k; \quad G(k) := \frac{3^k}{2^k - 1}; \quad I(k) := 2^k - 2 + [E(k)]; \quad T(k) := \frac{3^k - 1}{2^k - 1}.$$

A well-known result (see [1]) is

THEOREM 1: Let $k \neq 4$. Then the inequality

$$2^k . \{E(k)\} + [E(k)] \leq 2^k \tag{1}$$

holds iff $g(k) = I(k)$.

In [2] a very interesting hypothesis was proposed:

HYPOTHESIS 1: If $k \geq 3$, then the equality

$$[E(k)] = [G(k)] \tag{2}$$

holds.

In the present paper I establish the important result:

THEOREM 2: If Hypothesis 1 is true, then for every $k \neq 4$ and $k \geq 3$, $g(k) = I(k)$.

For $k \geq 3$ I need (when it is possible) the inequalities:

(a) $$\{G(k)\} > \frac{1}{2^k - 1};$$

(b) $$T(k) > [G(k)];$$

(c) $$E(k) > [G(k)];$$

and the important condition

(d) $T(k)$ is not integer.

The main result in the paper is

THEOREM 3: Let $k \geq 3$ be fixed. Then (2) holds iff (1) and (d) hold simultaneously.

Of course, if (d) is independently proved, then (1) and (2) are equivalent to each other. Moreover, in this case Hypothesis 1 is equivalent to the following statement:

"For every $k \geq 3$ and $k \neq 4$, the inequality (1) holds."

Therefore, if

HYPOTHESIS 2: "For every $k \geq 2$ the number $T(k)$ is not integer" is proved, then Hypothesis 1 is equivalent to the statement:

"For every $k \geq 3$ and $k \neq 4$, $g(k) = I(k)$."

Theorems 4, 5, 6 in the paper give the key to (2) (i.e. to Hypothesis 1). Finally, an open problem connected with Hypothesis 2 is posed.

LEMMA 1: If $k \geq 3$, then $G(k)$ is not an integer.

Proof: Let $G(k)$ be an integer for some $k \geq 3$. The equality $3^k = G(k).(2^k - 1)$ implies $3^k \equiv 0 \pmod{G(k)}$. Therefore, for some $m \in \{0, 1, 2, \dots, k\}$ it is satisfied $3^m = 2^k - 1$. But this equation has only the solutions: $m = 0$, $k = 1$ and $m = 1$, $k = 2$ (see [3]) which contradicts $k \geq 3$.

Further the answer to the question: "when it is possible to have $[E(k)] \neq [G(k)]$" is given.

THEOREM 4: Let $k \geq 3$ be fixed. Then the equality (2) is not true iff there exist three positive integers s, a, b, such that: $[\log_3 (2^{k+1} - 1)] < s \leq k$; a is odd; $a < b$; $HCF(a, b) = 1$, and the equality

$$3^s = 2^k.b - a$$

holds.

(Here and further $HCF(a, b)$ means the highest common factor of a and b.)

Proof: Under the conclusion of Lemma 1, (2) is not true iff there exists an integer $L \in (E(k), G(k))$. But a bijection between the last open interval and $(0, 1)$ exists. Therefore, $L = \dfrac{3^k}{2^k - x}$, where $x = \dfrac{a}{b} \in (0, 1)$ is a rational number and $HCF(a, b) = 1$. Hence

$$3^k = 2^k.L - \frac{a}{b}.L.$$

The last equality yields $L = d.b$ and $d = 3^{k-s}$, where $s \in \{0, 1, \dots, k\}$. Therefore $3^k = 2^k b - a$. Hence a is odd. Since $\frac{a}{b} \in (0, 1)$, the inequality $a < b$ holds. Then the last equality implies

$$3^s > 2^k.(a + 1) - a = 2^k + a.(2^k - 1) \geq 2^k + 2^k - 1 = 2^{k+1} - 1.$$

Therefore $[\log_3 (2^{k+1} - 1)] < s \leq k$ and the theorem is proved.

REMARK 1: It is easily seen that $a \equiv 3 \pmod 8$, or $a \equiv 7 \pmod 8$.

In the first case s is odd, but in the second case it is even.

For b the condition $b \in (\frac{3^s}{2^k}, \frac{3^s}{2^k - 1})$ is satisfied.

Let me substitute $a = b - t$, $t \in \{1, 2, \ldots, [\frac{3^s}{2^k}]\}$. Using that $b = [\frac{3^s}{2^k}] + 1$ and $b = \frac{3^s - t}{2^k - 1}$, I arrive to the following equivalent form of Theorem 4:

THEOREM 5: Let $k \geq 3$ be fixed. Then the equality (2) holds iff for every s, such that $1 + [\log_3(2^{k+1} - 1)] \leq s \leq k$, each number of the kind $A(s, t) \equiv \frac{3^s - t}{2^k - 1}$ is not integer when $t \in \{1, 2, \ldots, [\frac{3^s}{2^k}]\}$ and $HCF([\frac{3^s}{2^k}] + 1, [\frac{3^s}{2^k}] + 1 - t) = 1$.

Now, let me substitute $l = b - t$. When t runs the set $\{1, 2, \ldots, [\frac{3^s}{2^k}]\}$, l runs the same one. Using that $b = [\frac{3^s}{2^k}] + 1$ and $b = \frac{3^s + l}{2^k}$ I obtain another equivalent form of Theorem 4.

THEOREM 6: Let $k \geq 3$ be fixed. Then the equality (2) holds iff for every s, such that $1 + [\log_3 (2^{k+1} - 1)] \leq s \leq k$, each number of the kind $B(s, l) \equiv \frac{3^s + l}{2^k}$ is not integer when $l \in \{1, 2, \ldots, [\frac{3^s}{2^k}]\}$ and $HCF([\frac{3^s}{2^k}] + 1, l) = 1$.

In order to prove Theorem 2 and Theorem 3, I need some preliminary results.

LEMMA 2: Let x, y, v, u be integers and the first three of them be positive. Let the relations: $y < x$; $v \leq y - 1$; $x = u.y + v$, hold too. Then the inequality $[\frac{x + 1}{y}] < \frac{x}{y}$ holds iff $v < y - 1$.

Proof: The conditions of the Lemma imply $\frac{x + 1}{y} = u + \frac{v + 1}{y}$, 1. Let $v < y - 1$. Then $0 < \frac{v + 1}{y} < 1$. Therefore $[\frac{x + 1}{y}] = u$. But $u < u + \frac{v}{y}$, i.e. $[\frac{x + 1}{y}] < \frac{x}{y}$.

2. Let $v = y - 1$. Then: $\frac{x}{y} = u + 1 - \frac{1}{y}$; $[\frac{x+1}{y}] = u + 1$. Hence $\frac{x}{y} < [\frac{x+1}{y}]$. Therefore, the inequality $[\frac{x+1}{y}] < \frac{x}{y}$ implies $v < y - 1$.

The Lemma is proved.

LEMMA 3: Let $k \geq 3$ be fixed. Each one of (a) and (b) is equivalent to (d).

Proof: It is obvious that (a) and (b) are equivalent to each other. What is necessary now is to prove the equivalence of (b) and (d).

1. Let $T(k)$ be integer. Then the inequality $G(k) \geq T(k)$ yields:

$$[G(k)] \geq [T(k)] = T(k).$$

Therefore, (b) implies (d).

2. Let (d) hold. I substitute $x = 3^k - 1$; $y = 2^k - 1$, and consider the equality $x = u.y + v$, where u and v are integers. The assumption $v = y - 1$ implies $u = G(k)$, which contradicts Lemma 1. Therefore, $v < y - 1$. The assumption $v = 0$ implies $u = x/y = T(k)$. But the last equality contradicts (d). Therefore $0 < v < y-1$ and Lemma 2 is applicable. Hence, $[\frac{x+1}{y}] < \frac{x}{y}$, which under the above substitutions coincides with (b).

The lemma is proved.

Now, I am able to prove Theorem 3.

Proof of Theorem 3: After some elementary computations, (1) takes the equivalent form:

$$[G(k)] - [E(k)] \leq \frac{2^k}{2^k - 1} - \{G(k)\}. \qquad (3)$$

1. Let me prove that (2) implies (1) and (d).

If (2) holds, then to prove (3) (i.e., (1)) it is enough to verify the inequality $\{G(k)\} \leq \frac{2^k}{2^k - 1}$. The last one is obvious, because the left-hand side is less, but the right-hand side is greater than 1. So, (2) implies (1).

Since (2) holds Theorem 5 yields $T(k) = A(k, 1)$ is not integer, i.e., (2) implies (d), too.

2. Let me prove that (1) and (d) imply (2).

First, I rewrite (1), using (3), in the equivalent form

$$[G(k)] - [E(k)] \leq 1 - (\{G(k)\} - \frac{1}{2^k - 1}). \qquad (4)$$

If (d) holds, then (a) holds too (see Lemma 3). Therefore, $(\{G(k)\} - \frac{1}{2^k - 1}) > 0$ and (4) yields $[G(k)] - [E(k)] < 1$.

The last inequality implies immediatelly: $[G(k)] = [E(k)]$, i.e. (2). The Theorem is proved.

It is clear that Theorem 2 is a corollary of Theorem 1 and Theorem 3.

REMARK 2: For a fixed $k \geq 3$, (2) is equivalent to (c).

The proof of this statement is obvious so it can be omitted.

A particular case of Hypothesis 2 is the following result.

THEOREM 7: If k is even then $T(k)$ is not integer. If $k > 1$ is odd, then $T(k)$ is not integer in each of the cases: $k \equiv 3, 7, 9, 11, 15 \pmod{16}$.

Proof: 1. Let k be even. Then $2^k - 1 \equiv 0 \pmod 3$, but $3^k - 1 \equiv -1 \pmod 3$. Therefore, $T(k)$ is not integer.

2. Let $k > 1$ be odd and let k satisfy at least one of the congruences: $k \equiv 3, 7, 9, 11, 15 \pmod{16}$. In this case, $2^k - 1 \equiv 0 \pmod 7$, but it is impossible to have $3^k - 1 \equiv 0 \pmod 7$. Hence $T(k)$ is not integer.

The Theorem is proved.

Now, to prove Hypothesis 2 there only remains to solve the following

OPEN PROBLEM: Let $k > 1$ be odd. Prove that $T(k)$ is not integer in each one of the cases $k \equiv 1, 5, 13 \pmod{16}$.

REFERENCES:

[1] Vaughan R. C., The Hardy-Littlewood Method, Cambridge, 1981.

[2] Kudrevatov G., Number Theory Problems Workbook, Moscow, Prosvestenie, 1970, p. 59, Problem 13 (in Russian).

[3] Paskalev G., Penchev P., Problems from Mathematical Competitions, Sofia, Narodna Prosveta, 1983, p. 121, Problem 155 (in Bulgarian).