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INTEGER CLASS PROPERTIES ASSOCIATED WITH AN INTEGER MATRIX

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Abstract

This paper displays some old results in a new way and extends them in the context of the modular ring $\mathbb{Z}_6$. Various diophantine properties of an integer matrix modulo 6 are developed in a natural way from tables of the basic binary operations.

1. INTRODUCTION

We define here an integer matrix. This is defined naturally modulo 6 by $6r \pm i$, $i = 0,1,2,3,;$ $r = 0,1,2,\ldots$ Various diophantine properties are considered for the equivalence classes partitioned by $\mathbb{Z}_6$ [1]:

$$\overline{1} = \{4,10,16,22,\ldots\}, \quad \overline{4} = \{1,7,13,19,\ldots\},$$
$$\overline{2} = \{5,11,17,23,\ldots\}, \quad \overline{5} = \{2,8,14,20,\ldots\},$$
$$\overline{3} = \{6,12,18,24,\ldots\}, \quad \overline{6} = \{3,9,15,21,\ldots\}.$$

The elements of $\{\overline{1},\overline{2},\overline{3},\overline{4},\overline{5},\overline{6}\}$ are set out in Table 1 where they are defined in terms of the natural number r which defines the rows of the matrix M. As is well known the primes, $p > 3$, are defined in term of $6r \pm 1$, and so $p \in \{\overline{2},\overline{4}\}$.

column	1	2	3	4	5	6
row, r	$6r-2$	$6r-1$	$6r$	$6r+1$	$6r+2$	$6r+3$
0				1	2	3
1	4	5	6	7	8	9
2	10	11	12	13	14	15
3	16	17	18	19	20	21
4	22	23	24	25	26	27
5	28	29	30	31	32	33
6	34	35	36	37	38	39
7	40	41	42	43	44	45
8	46	47	48	49	50	51
9	52	53	54	55	56	57
10	58	59	60	61	62	63
11	64	65	66	67	68	69
12	70	71	72	73	74	75
13	76	77	78	79	80	81
14	82	83	84	85	86	87
15	88	89	90	91	92	93

Table 1

Similarly it is readily observed that $6|\overline{3}$ and $3|\overline{6}$; $2|\overline{1},\overline{3},\overline{5}$; $\overline{1}_r|(\overline{3}_r+\overline{5}_{r-1})$; $\overline{2}_r|(\overline{5}_r+\overline{5}_{r-1})$,

in which a_r represents an element in the rth row of M. Similarly we note the basic operations of addition and subtraction in Table 2(a).

$a \backslash b$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$
$\overline{1}$	$\overline{5}$	$\overline{6}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$
$\overline{2}$	$\overline{6}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
$\overline{3}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$
$\overline{4}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$	$\overline{1}$
$\overline{5}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$	$\overline{1}$	$\overline{2}$
$\overline{6}$	$\overline{4}$	$\overline{5}$	$\overline{6}$	$\overline{1}$	$\overline{2}$	$\overline{3}$

Table 2(a) $a+b, \quad b+a$

$a \backslash b$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$
$\overline{1}$	$\overline{1}$	$\overline{5}$	$\overline{3}$	$\overline{1}$	$\overline{5}$	$\overline{3}$
$\overline{2}$	$\overline{5}$	$\overline{4}$	$\overline{3}$	$\overline{2}$	$\overline{1}$	$\overline{6}$
$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{3}$
$\overline{4}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$
$\overline{5}$	$\overline{5}$	$\overline{1}$	$\overline{3}$	$\overline{5}$	$\overline{1}$	$\overline{3}$
$\overline{6}$	$\overline{3}$	$\overline{6}$	$\overline{3}$	$\overline{6}$	$\overline{3}$	$\overline{6}$

Table 2(b) $a\times b, \quad b\times a$

$a \backslash b$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$
$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$
$\overline{2}$	$\overline{4}$	$\overline{2}$	$\overline{4}$	$\overline{2}$	$\overline{4}$	$\overline{2}$
$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{3}$
$\overline{4}$	$\overline{4}$	$\overline{4}$	$\overline{4}$	$\overline{4}$	$\overline{4}$	$\overline{4}$
$\overline{5}$	$\overline{1}$	$\overline{5}$	$\overline{1}$	$\overline{5}$	$\overline{1}$	$\overline{5}$
$\overline{6}$	$\overline{6}$	$\overline{6}$	$\overline{6}$	$\overline{6}$	$\overline{6}$	$\overline{6}$

Table 2(c) $a*b \quad (a^b)$

$$\left.\begin{array}{ll} z = x +_6 y +_6 3, & x+y \le 3, \\ z = x +_6 y -_6 3, & 3 < x+y < 10, \\ z = x +_6 y -_6 9, & x+y \ge 10, \end{array}\right\} \qquad (1.1)$$

$$\left.\begin{array}{ll} z = x -_6 y +_6 9, & x - y < -2, \\ z = x -_6 y +_6 3, & -2 \le x - y < 4, \\ z = x -_6 y -_6 3, & x - y \ge 4. \end{array}\right\} \quad (1.2)$$

As expected, we have an abelian additive group in Table 2(a). Multiplication and exponentiation tables are displayed in Table 2(b) and (c). We can see that:

$$\text{there is no } a\text{: } a^2 \in \{\bar{2},\bar{5}\} \quad (1.3)$$
$$\text{for all } a, \quad a^3 = a. \quad (1.4)$$

This applies generally to even exponents (follow (1.3)) and odd exponents (follow (1.4)). We utilise these properties now to consider associated diophantine equations.

2. PYTHAGOREAN TRIPLES

We here relate Pythagorean triples to *M*. The only solutions $\{c,b,a\} \in \mathbb{Z}_6$ of:

$$c^2 = a^2 +_6 b^2 \quad (2.1)$$

are $\{\bar{4},\bar{1},\bar{6}\}$, $\{\bar{4},\bar{6},\bar{1}\}$, $\{\bar{4},\bar{3},\bar{4}\}$ and $\{\bar{4},\bar{4},\bar{3}\}$ because of the restrictive distribution of the squares.

$r_k \in \mathbb{N}$ is the row number of *M* which corresponds to $\bar{k} \in \{1,2,\ldots,6\}$. In general then

$$(f(r)_c)^2 = (f(r)_b)^2 +_6 (f(r)_a)^2 \quad (2.2)$$

where $f(r) = 6r_k \pm i$. Thus, for example $\{\bar{2},\bar{1},\bar{6}\}$ gives:

$$(6r_2 - 1)^2 = (6r_1 - 2)^2 +_6 (6r_6 + 3)^2 \quad (2.3)$$

To conform to the primitive Pythagorean triple grid defined previously [3,4] which is notationally convenient, we use the two internal parameters z and y and the counter $j = 1,2,3,\ldots$, and take $c > b > a$. These parameters are defined by:

$$z = c - b \quad (2.4)$$
$$y = b - a. \quad (2.5)$$

Case 1. z odd: The component c is given by [3]

$$c = j^2 + (j + z^{\frac{1}{2}})^2 \quad (2.6)$$

and

$$y = 2j^2 - z \quad (2.7)$$

with $j > (z/2)^{\frac{1}{2}}$ and $z = (2t-1)^2, t = 1,2,3\ldots$
From equations (2.4) and (2.5)

$$z = f(r)_c -_6 f(r)_b \quad (2.8)$$

and

$$y = f(r)_b -_6 f(r)_a \quad (2.9)$$

Thus, the relationships between the *r* parameters and *j* and *z* are easily established.

For $\{\overline{2},\overline{1},\overline{6}\}$

$$r_2 = (j^2 + 1 + (j + z^{\frac{1}{2}})^2)/6 \tag{2.10}$$

and if $z > 1$ then $j|z$ or $z|j$ are invalid j for primitive triples [3].

$$\left.\begin{aligned} r_1 &= r_2 - (z-1)/6 \\ r_6 &= r_2 - (j^2+2)/3 \end{aligned}\right\} \tag{2.11}$$

Using equations (2.2) and (2.11) we get

$$r_1 = (3r_6(r_6+1) - R)/z \tag{2.12}$$

with $R = (z^2 - 4z - 9)/12$ and $r_1 > r_6$.

For $\{\overline{2},\overline{5},\overline{6}\}$: r_2 and r_6 are the same as for equations (2.10), (2.11) respectively, and

$$r_5 = r_2 - (z+3)/6. \tag{2.13}$$

For $\{\overline{4},\overline{3},\overline{2}\}$:

$$\left.\begin{aligned} r_4 &= (j^2 - 1 + (j + z^{\frac{1}{2}})^2)/6 \\ r_3 &= r_4 - (z-1)/6 \\ r_2 &= r_4 - (j^2-1)/3 \end{aligned}\right\} \tag{2.14}$$

For $\{\overline{4},\overline{3},\overline{4}\}$: r_3 and r_4 and the same as for equation (2.14)

$$r_4' = r_4 - \frac{j^2}{3} \tag{2.15}$$

For the last three sets, r_k as a function of z can be derived as for equation (2.12) Examples are displayed in Table 3(a) where the permissible r functions, calculated from the above equations, are summarised. If z is prime to 3 it will not apply to equation (2.13) whereas z must be prime to 3 for the other column sets.

Case 2 z even: When z is even, the parities of b and a are opposite to those for z odd. Thus, instead of $\overline{2},\overline{1},\overline{6}$ for the component columns, we get $\overline{2},\overline{6},\overline{1}$ for c, b and a respectively. The same analysis as for z odd gives the results shown in Table 3(b)

Columns											
c^2	b^2	a^2	c	b	a	z	r			j	Triples
			$\overline{2}$	$\overline{1}$	$\overline{6}$		r_2	r_1	r_6		
						1	1	1	0	1	5,4,3
						25	63	59	22	11	377,352,135,
						121	101	81	60	11	605,484,363
$\overline{4}$	$\overline{1}$	$\overline{6}$									
			$\overline{2}$	$\overline{5}$	$\overline{6}$		r_2	r_5	r_6		
						9	11	9	5	4	65,56,33
						81	51	37	34	7	305,224,207
						225	219	181	122	17	1313,1088,735
			$\overline{4}$	$\overline{3}$	$\overline{2}$		r_4	r_3	r_2		
						1	2	2	1	2	13,12,5
						49	28	20	20	5	169,120,119
						361	214	154	149	14	1285,924,893
$\overline{4}$	$\overline{3}$	$\overline{4}$									
			$\overline{4}$	$\overline{3}$	$\overline{4}$		r_4	r_3	r'_4		
						1	4	4	1	3	25,24,7
						49	34	26	22	6	205,156,133
						361	230	170	155	15	1381,1020,931

Table 3(a) Primitive Pythagorean triples with z odd

Columns											
c^2	b^2	a^2	c	b	a	z	r			j	Triples
			$\overline{2}$	$\overline{6}$	$\overline{1}$		r_2	r_6	r_1		
						2	11	10	3	4	65,63,16
						2	33	32	5	7	197,195,28
						32	23	17	15	4	137,105,88
$\overline{4}$	$\overline{6}$	$\overline{1}$									
			$\overline{2}$	$\overline{6}$	$\overline{5}$		r_2	r_5	r_6		
						2	3	2	1	2	17,15,8
						8	5	3	3	2	29,21,20
						98	89	72	51	8	533,435,308
			$\overline{4}$	$\overline{2}$	$\overline{3}$		r_4	r_2	r_3		
						2	6	6	2	3	37,35,12
						2	24	24	4	6	145,143,24
						8	14	13	6	4	85,77,36
						50	58	50	30	7	349,299,180
$\overline{4}$	$\overline{4}$	$\overline{3}$									
			$\overline{4}$	$\overline{4}$	$\overline{3}$		r_4	r'_4	r_3		
						18	12	9	8	3	73,55,48
						72	54	42	34	6	325,253,204
						288	164	116	116	9	985,697,696

Table 3(b) Primitive Pythagorean Triples with z even

The triple component c is given by [3]

$$c = (z/2) + ((z/2)^{\frac{1}{2}} + (2j-1))^2 \qquad (2.16)$$

and $(b-a)$ or y is given by:

$$y = (2j-1)^2 - z \qquad (2.17)$$

with $j > (z^{\frac{1}{2}} + 1)/2$ and $z = 2t^2$.

For $\{\overline{2},\overline{6},\overline{1}\}$:

$$\left.\begin{aligned} r_2 &= \left\{(z/2+1) + ((2j-1) + (z/2)^{\frac{1}{2}})^2\right\}/6 \\ r_6 &= r_2 - (z+4)/6 \\ r_1 &= r_2 - 2j(j-1)/3 \end{aligned}\right\} \qquad (2.18)$$

For $\{\overline{2},\overline{6},\overline{5}\}$:

r_2 and r_6 as for equation (2.18)

$$r_5 = r_2 - 2(j^2 - j + 1)/3 \qquad (2.19)$$

For $\{\overline{4},\overline{2},\overline{3}\}$:

$$\left.\begin{aligned} r_4 &= \left\{(z/2-1) + ((2j-1) + (z/2)^{\frac{1}{2}})^2\right\}/6 \\ r_2 &= r_4 - (z-2)/6 \\ r_3 &= r_4 - 2j(j-1)/3 \end{aligned}\right\} \qquad (2.20)$$

For $\{\overline{4},\overline{4},\overline{3}\}$:

r_3 and r_4 as for equation (2.20)

$$r_4' = r_4 - z/6\,. \qquad (2.21)$$

This column set only applies to z values not prime to 6.

3. CONCLUSIONS

A theorem of Fermat [2] shows that for prime p, $(N,p) = 1$, and $k_N \in \mathbb{Z}$,

$$N^{p-1} = 1 + pk_N \qquad (3.1)$$

Then for $a^2 = 1 + 3k_a$, $b^2 = 1 + 3k_b$, and $c^2 = 1 + 3k_c$, (3.1) implies that

$$(k_c - k_b) = (1/3) + k_a \qquad (3.2)$$

which gives a non-integer solution. However, if one of the components is not prime to p (i.e. has a factor of 3 in the case above), then an integer solution is possible.

Suppose $a = 3^w A$, where w is a positive integer and A is prime to 3, then we get

$$a^2 = 3^{2w} + 3^{2w+1} k_A \qquad (3.3)$$

This result explain why the column sets for the components are confined to $\{\overline{4},\overline{3},\overline{4}\}$ or $\{\overline{4},\overline{3},\overline{2}\}$ and $\{\overline{2},\overline{1},\overline{6}\}$ or $\{\overline{2},\overline{5},\overline{6}\}$ which give integer solutions for the Pythagorean triples. Columns $\overline{3}$ and $\overline{6}$ both contain numbers with 3 as a factor; (see (1.2)). For example, equation (3.3) applies in the case of $\{\overline{2},\overline{1},\overline{6}\}$. Thus, for the triple (5,4,3) where $w = 1$, $k_A = 0$, $k_c = 8$ and $k_b = 5$. Whilst for the column set $\{\overline{2},\overline{5},\overline{6}\}$ and the triple (305,224,207), $w = 2$, $k_A = 176$, $k_c = 31008$ and $k_b = 16725$.

The cubic powers are listed in Table 2(c) since $a^3 = a \in \{\overline{1},\overline{2},\overline{3},\overline{4},\overline{5},\overline{6}\}$,
If $c = d^{3/2}, b = e^{3/2}$ and $a = f^{3/2}$, with c, b and a being integer components of a Pythagorean triple, then

$$d^3 = e^3 +_6 f^3 \qquad (3.4)$$

Integer values of $d^{3/2}, e^{3/2}$ and $f^{3/2}$ only occur when $d = d'^2, e = e'^2$ and $f = f'^2; d', e', f' \in \mathbb{Z}$. Thus $d^{3/2} = d'^3$, and so on. It is known that [2]

$$d^6 \neq e^6 + f^6 \qquad (3.5)$$

and since $(d^{3/2})^2 = d'^6$, $(e^{3/2})^2 = e'^6$ and $(f^{3/2})^2 = f'^6$ equation (3.4) cannot be valid for this case. The same argument can be applied when $n = 5,7$. A more general analysis to include non-integer values of $d^{3/2}$ and all odd and even exponents is to be given in a following paper.

REFERENCES

1. A. P. Hillman and G. L. Alexanderson, *A First Undergraduate Course in Abstract Algebra*, Wadsworth, Belmont, 1973.

2. J. Hunter, *Number Theory*, Oliver and Boyd, Edinburgh, 1964.

3. J. V. Leyendekkers and J. M. Rybak The generation and analysis of Pythagorean triples within a two-parameter grid. *International Journal of Mathematical Education in Science and Technology,* in press.

4. J. M. Rybak, J. V Leyendekkers and A. G. Shannon, Recurrence relation analysis of Pythagorean triple patterns. *Notes on Number Theory and Discrete Mathematics*, 1, 1, 1995, 1-10.