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# FIXED POINTS OF CERTAIN DIVISOR FUNCTION

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Let $\sigma(n)$ be the sum of the positive divisors of n, $L(n)$ be the sum of the aliquot parts of n, i.e. the positive divisors of n other than n itself, so that $L(n) = \sigma(n) - n$. Perfect, amicable and sociable numbers are fixed points of the arithmetic function L and its iterates. Let L+ and L- be defined by $L\pm(n) = L(n) \pm 1$. Jerrard and Temperley [3] studied the existence of fixed points of L+ and L-. The fixed points of $L+(n)$ are the almost perfect numbers and the fixed points of $L-(n)$ are the quasi-perfect numbers [2].

If d divides n and $(d, n/d) = 1$, call d a unitary divisor of n, $\sigma*(n)$ the sum of unitary divisors of n. Beck and Najar [1] introduced the arithmetic functions $L^*\pm(n) = \sigma*(n) - n \pm 1$ and proved that $L^*-$ has no fixed points.

A divisor $d > 0$ of the positive number n is called bi-unitary if the greatest common unitary divisor of d and $n/d$ is 1. Let $\sigma^{**}(n)$ be the sum of the bi-unitary divisors of n. For $n = p^\alpha$

$$\sigma^{**}(n) = \begin{cases} 1 + p + \dots + p^{2\alpha} - p^\alpha, & \text{if } n = p^{2\alpha} \\ 1 + p + \dots + p^{2\alpha+1}, & \text{if } n = p^{2\alpha+1} \end{cases} \tag{1}$$

and $\sigma^{**}(n)$ is a multiplicative function.

Let us define the new arithmetic functions

$$L^{**}(n) = \sigma^{**}(n) - n$$

$$L-^{**}(n) = L^{**}(n) - 1$$

We prove that $L-^{**}(n)$ has no fixed points.

<u>LEMMA 1</u>: For natural number n, $L-^{**}(n) = 0$ <u>iff</u> $n = p$ or $n^2 = p$ (where p is a prime number).

Proof: Let $n = p_1^{\alpha_1} p_2^{\alpha_2} k$, where $(k, p_1p_2) = 1$. Then (1) implies:

$$\sigma^{**}(n) \geq (1 + p_1^{\alpha_1})(1 + p_2^{\alpha_2})\sigma^{**}(k) \geq (1 + p_1^{\alpha_1})(1 + p_2^{\alpha_2})k > k + n,$$

because $\sigma^{**}(k) \geq k$. Then:

$$L-^{**}(n) = \sigma^{**}(n) - n - 1 > k - 1 \geq 0.$$

it can be seen directly that if $n = p$, then $L^{**}_{-}(n) = 0$ and, if $n = p^2$, then $L^{**}_{-}(n) = 0$, too, while, if $n = p^{\alpha}$ for $\alpha > 2$, then

$$L^{**}_{-}(n) = \sigma^{**}(n) - n - 1 = \begin{cases} p + \dots + p^{2\beta} - p^{\beta} > 0, & \text{if } n = p^{2\beta} \\ p + \dots + p^{2\beta+1} > 0 & , \text{if } n = p^{2\beta+1} \end{cases}$$

<u>LEMMA 2</u>: For natural number $n$:

(a) if $n = 1$, then $L^{**}_{-}(n) = -1$

(b) if $n > 1$, $n \equiv 1 \pmod 2$, then $L^{**}_{-}(n) \equiv 0 \pmod 2$

(c) if $n = 2^{\alpha}$ for $\alpha > 0$, then $L^{**}_{-}(n) \equiv 0 \pmod 2$

(d) if $n \neq 2^{\alpha}$, $n \equiv 0 \pmod 2$, then $L^{**}_{-}(n) \equiv 1 \pmod 2$

Proof: Lemma follows by

$$\sigma^{**}(n) = \begin{cases} 1 + p + \dots + p^{2k} - p^{k} \equiv 2\alpha \equiv 0 (\mathrm{mod}\ 2), & \text{if } p>2,\ \alpha = 2k>0 \\ 1 + p + \dots + p^{2k+1} \equiv 0 (\mathrm{mod}\ 2), & \text{if } p > 2,\ \alpha = 2k+1 \\ 1 + p + \dots + p^{2k} - p^{k} \equiv 1 (\mathrm{mod}\ 2), & \text{if } p = 2,\ \alpha = 2k > 0 \\ 1 + p + \dots + p^{2k+1} \equiv 1 (\mathrm{mod}\ 2), & \text{if } p = 2,\ \alpha = 2k+1 \end{cases}$$

and the multiplicativity of $\sigma^{**}(n)$.

<u>THEOREM</u>: $L^{**}_{-}(n)$ has no fixed points, i.e., $L^{**}_{-}(n) = n$ has no solutions in natural numbers.

Proof: By Lemma 2 it follows that $L^{**}_{-}(n) \equiv n + 1 (\mathrm{mod}\ 2)$, if $n > 1$, $n \neq 2^{\alpha}$ which implies that $L^{**}_{-}(n) = n$ has no solutions. Let $n = 2^{\alpha}$. By Lemma 1, if $\alpha = 1$ or $2$, $L^{**}_{-}(n) = 0$. If $\alpha > 2$:

$$L^{**}_{-}(n) = \sigma^{**}(n) - n - 1 = \begin{cases} 2(1+2+\dots+2^{2k-2}+2^{k}+\dots+2^{2n-2}), & \text{if } \alpha=2k \\ 2(1+2+\dots+2^{2k-1}) & , \text{if } \alpha=2k+1 \end{cases}$$

which implies that $L^{**}_{-}(n) \equiv 0 \pmod 2$, but $n = 2^{\alpha}$, $\alpha > 2$ implies that $L^{**}_{-}(n) = n$ has no solutions.

REFERENCES:


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[3] <u>R. Jerrard, N. Temperley</u>, Almost perfect numbers, Math. Mag. Vol. 46, 1973, 84-87.