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RECURRENCE RELATION ANALYSIS OF PYTHAGOREAN TRIPLE PATTERNS

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ABSTRACT

This paper explores recurrence relations in their role of providing internal generators of Pythagorean triples. While the relation of Pellian recurrence relations to diophantine equations in general is not new, this paper classifies the internal generators according to their parity and primality.

1. INTRODUCTION

Pythagorean triples (PTs) a, b, c

$$a^2 + b^2 = c^2 \tag{1.1}$$

may be characterized in terms of the two internal parameters z and y by

$$z = c - b \tag{1.2}$$

$$y = b - a \tag{1.3}$$

so that [1]

$$a = z \pm G^{1/2} \tag{1.4}$$

$$b = z + y \pm G^{1/2} \tag{1.5}$$

$$c = (2z + y) \pm G^{1/2} \tag{1.6}$$

in which $G = 2z(y + z)$.

Unlike z, the parameter y is always odd, since b ***and*** **a** are of opposite parity. It has been shown [2] that triples with different parity z, but with a common y, are linked via Pellian sequences. This use of internal generators and Pellian equations extends the work of Shannon and Horadam [3, 4].

In the present paper the aim is to analyse the structure of the y parameter, which may be a prime or a composite number. Equations will be derived to predict those values of composite y containing a given prime. The prime factor patterns of y are compared with those of a composite number equal to y but derived from a function independent of PTs. This gives y and the associated parameters a direct link with more general parameters that determine primality. Finally, a consideration of the end digit patterns for y and z shows how simple the structure can become when viewed from this angle.

2. THE RECURRENCE RELATIONS

We shall see, for any given z, that the internal parameter y is generated by the second order nonhomogeneous linear recurrence relation [1]

$$y_n = 2y_{n-1} - y_{n-2} + D \tag{2.1}$$

where

$$D = \begin{cases} 4 \ , & z \text{ odd} \\ 8 \ , & z \text{ even,} \end{cases}$$

and

$$z = \begin{cases} (2k-1)^2, & z \text{ odd ,} \\ 2k^2, & z \text{ even,} \end{cases} \quad k = 1, 2, 3 \ldots$$

The equation (2.1) has solutions

$$y_j = \begin{cases} 2j^2 - z \ , & \left(j > (z/2)^{\frac{1}{2}}\right), \quad z \text{ odd} & (2.2a) \\ (2j-1)^2 - z \ , & \left(j > \left(z^{\frac{1}{2}} + 1\right)/2\right), \quad z \text{ even} & (2.2b) \end{cases}$$

For example, Table 1 shows some cases for $z = 1, 2, 9$ and $j = 1, 2, \ldots, 7$. The presence of the

j_0 is the same as j corresponding to $y = p$ for a given z, as given by (2.2a, b). Hence

$$j_0 = \begin{cases} ((p+z)/2)^{\frac{1}{2}} & (z \text{ odd}) \quad (2.6a) \\ \left((p+z)^{\frac{1}{2}}+1\right)/2 & (z \text{ even}) \quad (2.6b) \end{cases}$$

These regular patterns depend on the contribution of all PTs, that is, all j and z which satisfy (2.2 a, b). For example, with $p = 7$ and $z = 1$:

$$j_0 = 2\,, r = 1; \quad j_1 = (rp - j_0) = 5; \quad j_2 = j_0 + np/2 = 9;$$

$$j_3 = j_1 + (n-1)p/2 = 12, \quad \text{so that } y_0 = 7, y_1 = 49, y_2 = 161, y_3 = 287 \text{and so on;}$$

all the numbers contain the prime factor 7.

3. CASE 1: z GENERATES $y = p$

As shown in Part 2, when $y = p$, j has the values given by (2.6a) or (2.6b) and is designated j_0. The sequence of composite y values containing p can then be generated directly from (2.2a) to (2.5b) as illustrated in the previous paragraph.

Composite values of y (with factor p) must follow the equation [1]

$$m = p^2 + 2p(s-1) \tag{3.1}$$

which applies generally to odd numbers that are not prime. The parameter s = 1, 2, 3, ... and has no formal structure within the odd composite number system. Therefore, it is of interest to see what values of s apply for $y = m$.

Applying the equations of Section 2 we find that, within the PT system, values of s are generated by the recurrence relation

$$s_n = s_{n-1} + (n + r - 1)v \qquad n > 1 \tag{3.2}$$

where $v = v_1$ for even n and $v = v_2$ for odd n, with

$$v_1 + v_2 = \begin{cases} p & (z \text{ odd}) \quad (3.3a) \\ 2p & (z \text{ even}) \quad (3.3b) \end{cases}$$

and

$$v_1 = \begin{cases} 2j_0 - (r-1)p & (z \text{ odd}) \quad (3,4a) \\ 2(2j_0 - (r-1)p - 1) & (z \text{ even}) \quad (3.4b) \end{cases}$$

s_1 is obtained from (3.1) using j_1 to calculate m_1 from (2.2a,b). Solutions of (3.2) are set out in Table 2.

z	n	s_n
odd	odd	$\left(r^2 + (n-1)r + \frac{1}{4}(n^2 - 2n - 1)\right)p - (2r + n - 1)j_0 + 1.5$
odd	even	$\frac{1}{2}(3 + 2j_0 n + (n^2 - 2)p/2)$
even	odd	$((2r + n - 1)^2 - 1)p/2 - (2j_0(2r + n - 1) - (2r + n)) + 0.5$
even	even	$(2j_0 - 1)n + (n^2 - 1)p/2 + 1.5$

Table 2. Solutions of recurrence relation (3.2)

For example, with $z = 1$ $p = 7$ and $n = 3$,

$r = 1; j_0 = ((p + z)/2)^{1/2} = 2$, so that $v_1 = 2j_0 = 4$,

$v_2 = p - v_1 = 3; j_1 = rp - j_0 = 5; m_1 = 2j_1^2 - z = 49; s_1 = (m_1 - p^2)/2p + 1 = 1;$

$s_3 = s_1 + 2v_1 + 3v_2 = 18$

Using Table 2, we see that

$$s_3 = (r^2 + (n-1)r + 1/4(n^2 - 2n - 1))p - (2r + n - 1)j_0 + 1.5 = 18$$

Thus $m_3 = 49 + 14 \times 17 = 287 = 7 \times 41$, from (3.1) which is in agreement with y_3 estimated in Section 2.

When $r = 1$ the values of s follow the pattern odd odd even even when z is odd, and when z is even s is always odd. For $r > 1$, apart from an initial perturbation, the same patterns are followed.

Some values of m and the associated parameters for a given z are shown in Table 3.

4. CASE 2: z DOES NOT GENERATE $y = p$

When p is not generated directly it will appear as a factor in some of the y values. The lowest value of y with p as a factor will equal pp_k where p_k is the lowest value prime that gives j as an integer in equation (2.1), that is

$$j_0 = \begin{cases} ((pp_k + z)/2)^{\frac{1}{2}} & (z \text{ odd}) \quad (4.1a) \\ \left((pp_k + z)^{\frac{1}{2}} + 1\right)/2 & (z \text{ even}) \quad (4.2b) \end{cases}$$

For example, if $z = 1$ and $p = 79$, the lowest value for p_k is 31 and $j_0 = 35$. Thereafter, the j and s functions are the same as for case 1. An exception is when $p = 7$ when j_1 should be calculated from equation (2.1).

z	1	32	225	80000	29929	72962
p	7	17	17	89	823	1567
j_0	2	4	11	142	124	137
r	1	1	2	4	1	1
v_1	4	14	5	32	248	546
m_1	49	697	833	104041	947273	8112359
	7×7	17×41	17×7×7	89×7×167	823×1151	1567×31×167
s_1	1	13	17	541	165	1806
m_2	161	1649	1343	132521	1763689	11534687
	7×23	17×97	17×79	89×1489	823×2143	1567×17×433
s_2	9	41	32	701	661	2898
m_3	287	3689	2975	288449	4603039	35867063
	7×41	17×7×31	17×7×5×5	89×7×463	823×7×17×47	1567×47×487
s_3	18	101	80	1577	2386	10662
m_{30}	22897	267257	141287	8640209	310921993	2235609127
	7×3271	17×79×199	17×8311	89×97081	823×17×71×313	1567×167×8543
s_{30}	1633	7853	4148	48497	188485	712558
m_{95}	223111	2640593	1351143	77272025	3101538871	22547652319
	7×31873	17×17×9137	17×9×8831	89×25×34729	823×17×31×7151	1567×23321×617
s_{95}	15934	77657	39732	434069	1883878	7193746

Table 3. Some values of s_n

5. CASE 3: $z = p^2$

In this case (which only occurs for z odd) the j and s functions are particularly simple:

$$j_n = (n+1)p \tag{5.1}$$

and

$$s_n = 1 + n(n+2)p \tag{5.2}$$

$n = 0, 1, 2...$ In these cases it is p^2 that is the factor of m, rather than p.

If $z = p^2$ and $y = p^2 p_k$ where p_k is another prime, are substituted into equations (1.4) to (1.6) we get

$$a = p^2(1 \pm H^{1/2}) \tag{5.3}$$

$$b = p^2(1 + p_k \pm H^{1/2}) \tag{5.4}$$

$$c = p^2(2 + p_k \pm H^{1/2}) \tag{5.5}$$

with $H = 2\,(1 + p_k)$, so that such triples are always non-primitive. The same applies when p_k is replaced by multiple primes. This is in accord with previous results [2] whereby y values for primitive triples were shown to have factors of only certain primes (p_k) such as 7, 17, 23, 31, 41, 47, 71.... The remaining primes (p_i) are 3, 5, 11, 13, 19, 29... and do not appear to be compatible with the z, j grid in respect to the y values. That is, these primes do not give integer solutions when $y = p$ for equation (2.1).

When z is odd and j even, $(z+y)$ should be divisible by 8 to conform to the equations, if j is odd then $(z_i + y) = 2z_k$ where z_k is odd.

When z is even, then $(y+z(\text{even})) = z(\text{odd})$ with $z(\text{odd}) \neq 1$. This applies to all j. The p_i primes do not conform to these criteria.

6. END DIGIT PATTERNS FOR y AND z

The ending (or last digit) of z will depend on the parity. For example, when z is odd the endings will be 1, 5 or 9, whereas when z is even the endings will be 0, 2 or 8. The last digit of y will depend on z (Table 4).

z	even	odd	even	odd	even	odd
z^*	0	1	2	5	8	9
y^*	9, 5, 1	9, 7, 1	9, 7, 3	7, 5, 3	7, 3, 1	9, 3, 1

Table 4: Last Digits of y, z ($z^* \equiv z \pmod{10}$)

If the m values with a given factor p are grouped according to the last digit, for a given z, then a very simple pattern emerges. The j values, here designated as k (since they are extracted from a j series), are generated from the recurrence relations (6.1) and (6.2). When

$$|m^* - x^*| = \begin{cases} 1 & z^* = 2,8 \\ 5 & z^* = 0 \\ 0 & z^* = 5 \\ 8 & z^* = 1,9 \end{cases}$$

in which $x^* \equiv x (\mathrm{mod} 10)$,

$$k_n = k_{n-2} + 5p, n \geq 3; \tag{6.1}$$

otherwise

$$k_n = k_{n-4} + 5p, n \geq 5. \tag{6.2}$$

The first four values of k are most simply obtained from (2.1) and are related by:

$$k_n = k_{n-2} + t \tag{6.3}$$

where $n = 3$ or 4. Some values of k_1, k_2 and t are listed in Table 5. For the $|m^* - x^*|$ listings above, t equals $5p$. When p is generated directly for a particular z (that is, (2.6 a or b) yields integer j_0) and the last digit of p is the same as that of m, t has different values for $n = 3$ and $n = 4$. This is because p itself starts the sequence of m but has to be discounted because it is a prime. Double values of t occur for certain digit endings when p is 7 but is not generated directly for a given z. This anomaly accords with Case 2 above. All values of t are divisible by 5 or p.

m^*	z	prime, p	7	17	23	31	41	47	71
9	2	k_1	6	6	21	20	50	51	30
		k_2	16	40	26	51	91	91	101
		t	14	40	69	85	65	94	225
7	2	k_1	9	12	44	12	9	44	42
		k_2	27	29	49	74	74	192	172
		t	25,10	45	23	70	123	188,47	142
3	2	k_1	13	23	113	43	33	98	113
		k_2	23	63	118	113	173	138	243
		t		t equals $5p$					
7	8	k_1	8	83	53	8	38	78	48
		k_2	28	88	63	148	168	158	308
		t		t equals $5p$					
3	8	k_1	15	15	30	55	45	31	95
		k_2	21	20	40	70	86	111	166
		t	20,15	51	46	31	75	94	95
1	8	k_1	7	32	7	24	79	17	24
		k_2	14	37	17	39	127	64	119
		t	15	17	92	93	123,82	155	213
9	1	k_1	5	20	55	35	70	20	65
		k_2	30	65	60	120	135	215	290
		t		t equals $5p$					
7	1	k_1	12	37	32	27	12	27	77
		k_2	23	48	37	58	53	67	148
		t	21,14	45,40	46	70	140	141	130
1	1	k_1	9	14	9	66	29	74	136
		k_2	16	31	14	89	94	114	219
		t	10	40	92	85,70	82	47	213,142
9	9	k_1	8	8	27	12	77	13	18
		k_2	13	42	42	43	87	107	53
		t	14	35	46	100	41	115	284
3	9	k_1	6	9	19	19	36	34	89
		k_2	29	26	96	74	46	81	124
		t	28,7	50	92,23	62	123	120	142
1	9	k_1	15	25	50	50	200	60	160
		k_2	20	60	65	105	210	175	195
		t		t equals $5p$					

Table 5: m^*: Last Digit of *m*

The way in which the *k* values arise can be established from (2.2a, b and Table 4). For example, when $z = 1$, *m* ends in 9 for every *j* ending in 5 or 0. However, not all these *m* values will have

p as a factor. In this case, every alternate value of m that ends in 9 and has p as a factor will occur in $5p$ jumps for j and hence k. On the other hand, with $z = 1$ and $m^* = 7$, j must end in 2, 3, 7 or 8. The j^* values fall into the pattern 2, 3, 3, 7, 7, 8, 8, 2, 2, ... so that $k_n - k_{n-4}$ gives the $5p$ increment rather than $k_n - k_{n-2}$ as for $m^* = 9$. Some examples of PTs for $m^* = 1$, $z = 1$, $p = 7$ are given in Table 6.

m	161	511	721	1351	3871
n	1	2	3	4	5
j	9	16	19	26	44
Source	Table 5	Table 5	(6.3)	(6.3)	(6.2)
Triple	181	545	761	1405	3961
	180	544	760	1404	3960
	19	33	39	53	89
	145	481	685	1301	3785
	144	480	684	1300	3784
	-17	-31	-37	-51	-87

Table 6. Triples coresponding to y = m
(m has the prime factor 7, with z = 1 and m* = 1;
triples are from (1.4) to (1.6))

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