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Leonardo's bivariate and complex polynomials

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Abstract: Given the purpose of mathematical evolution of Leonardo's sequence, we have the prospect of introducing complex polynomials, bivariate polynomials and bivariate polynomials around these numbers. Thus, this paper portrays in detail the insertion of the variable x, y and the imaginary unit i in the sequence of Leonardo. Nevertheless, the mathematical results from this process of complexification of these numbers are studied, correlating the mathematical evolution of that sequence.

Keywords: Leonardo complex bivariate polynomials, Leonardo polynomials, Leonardo sequence.
2020 Mathematics Subject Classification: 11B37, 11B69.

1 Introduction

Leonardo's sequence was initially presented by Catarino and Borges [5]. Historically, it is believed that these numbers have been studied by Leonardo de Pisa, known as Leonardo Fibonacci, and therefore not proven in any work in the literature, due to the scarcity of research [3]. This sequence has been studied and evolved mathematically, as we can see in the works of [2, 7–9].

Thus, we have the Leonardo sequence satisfying the following recurrence relationship:*

$$Le_n = Le_{n-1} + Le_{n-2} + 1, n \geq 2. \tag{1}$$

And yet, for $n+1$ one can rewrite this recurrence relationship as $Le_{n+1} = Le_n + Le_{n-1} + 1$. Also, subtracting $Le_n - Le_{n+1}$ gives another recurrence relation for this sequence.

$$\begin{aligned} Le_n - Le_{n+1} &= Le_{n-1} + Le_{n-2} + 1 - Le_n - Le_{n-1} - 1, \\ Le_{n+1} &= 2Le_n - Le_{n-2}, \end{aligned} \tag{2}$$

where $Le_0 = Le_1 = 1$ are the initial conditions.

Thus, the initial values of the sequence are as follows: $1, 1, 3, 5, 9, 15, 25, \ldots$.

In order to continue the mathematical evolutionary process of Leonardo's numbers, in this paper, we will present a study around Leonardo's numbers in their polynomial, bivariate polynomial and complex bivariate polynomial form.

We can find sequences in their polynomial form in works presented in the literature of pure mathematics, and yet, according to [6] the complex bivariate polynomials encompasses the polynomial terms of the studied sequence in an evolutionary process of its algebraic form. That is, first, polynomials are considered with one variable and two variables, then the imaginary component i is inserted, then these polynomials are explored in their complex form.

2 Leonardo's polynomials

Based on the Fibonacci polynomials, studied in 1883 by the mathematicians Eugène Catalan (1814–1894) and Ernst Erich Jacobsthal (1881–1965) [1], one can then introduce Leonardo's polynomials.

Definition 2.1. *Leonardo's polynomials, $l_n(x)$, for $n \geq 3$ are given by:*

$$l_n(x) = 2x l_{n-1}(x) - l_{n-3}(x),$$

with $l_0(x) = l_1(x) = 1$ and $l_2(x) = 3$.

The first terms of the sequence are given in the following Table 1.

* **Post-Publication Correction Note:** The notation used in Eq. (1) and (2) was corrected in the online version on 11 May 2022.

n	$l_n(x)$
0	1
1	1
2	3
3	$6x-1$
4	$12x^2-2x-1$
5	$24x^3-4x^2-2x-3$
6	$48x^4-8x^3-4x^2-12x+1$
$\vdots$	$\vdots$

Table 1. First terms of Leonardo's polynomial sequence

Theorem 2.2. *The matrix form of Leonardo's polynomials, for $n \geq 2$ and with $n \in \mathbf{N}$, is given by:*

$$\begin{bmatrix} 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2x & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}^n = \begin{bmatrix} l_{n+2}(x) & l_{n+1}(x) & l_n(x) \end{bmatrix}.$$

Proof. We use the principle of finite induction.

For $n = 2$, we have that:

$$\begin{bmatrix} 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2x & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 12x^2-2x-1 & 6x-1 & 3 \end{bmatrix} = \begin{bmatrix} l_4(x) & l_3(x) & l_2(x) \end{bmatrix}.$$

Validating equality.

Assuming it is valid for $n = k, k \in \mathbf{N}$, we have that:

$$\begin{bmatrix} 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2x & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}^k = \begin{bmatrix} l_{k+2}(x) & l_{k+1}(x) & l_k(x) \end{bmatrix}.$$

Now, verifying that it is valid for $n = k+1$, we have that:

$$\begin{aligned} \begin{bmatrix} 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2x & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}^k \begin{bmatrix} 2x & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} &= \begin{bmatrix} l_{k+2}(x) & l_{k+1}(x) & l_k(x) \end{bmatrix} \begin{bmatrix} 2x & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2xl_{k+2}(x) - l_k(x) & l_{k+2}(x) & l_{k+1}(x) \end{bmatrix} \\ &= \begin{bmatrix} l_{k+3}(x) & l_{k+2}(x) & l_{k+1}(x) \end{bmatrix}. \end{aligned}$$

□

The characteristic equation of this Leonardo polynomial sequence, is given by $t^3 - 2xt^2 + 1 = 0$, where x is the polynomial variable. So, we have to t_1, t_2 and t_3 are the roots of the characteristic equation.

Theorem 2.3. *Binet's formula of Leonardo's polynomials, with* $n \in \mathbf{Z}$*, is given by:*

$$l_n(x) = \alpha t_1^n + \beta t_2^n + \gamma t_3^n,$$

where t_1, t_2, t_3 *are the roots of the characteristic equation* $t^3 - 2xt^2 + 1 = 0$ *and*

$$\alpha = \frac{3 + (-t_2 - t_3) + t_2 t_3}{t_1^2 - t_1 t_2 - t_1 t_3 + t_2 t_3}, \ \beta = \frac{3 + (-t_1 - t_3) + t_1 t_3}{t_2^2 - t_2 t_3 - t_1 t_2 + t_1 t_3}, \ \gamma = \frac{3 + (-t_1 - t_2) + t_1 t_2}{t_3^2 + t_1 t_2 - t_1 t_3 - t_2 t_3}.$$

Proof. Through the Binet formula $l_n = \alpha t_1^n + \beta t_2^n + \gamma t_3^n$ and the recurrence of Leonardo's polynomials $l_n(x) = 2x l_{n-1}(x) - l_{n-3}(x)$, with the initial values $l_0(x) = l_1(x) = 1$ and $l_2(x) = 3$, it is possible to obtain the following system of equations:

$$\begin{cases} \alpha + \beta + \gamma & = 1 \\ \alpha t_1 + \beta t_2 + \gamma t_3 & = 1 \\ \alpha t_1^2 + \beta t_2^2 + \gamma t_3^2 & = 3 \end{cases}.$$

Solving the system, we have that:

$$\alpha = \frac{3 + (-t_2 - t_3) + t_2 t_3}{t_1^2 - t_1 t_2 - t_1 t_3 + t_2 t_3},$$
$$\beta = \frac{3 + (-t_1 - t_3) + t_1 t_3}{t_2^2 - t_2 t_3 - t_1 t_2 + t_1 t_3},$$
$$\gamma = \frac{3 + (-t_1 - t_2) + t_1 t_2}{t_3^2 + t_1 t_2 - t_1 t_3 - t_2 t_3}.$$

□

Theorem 2.4. *The generating function of Leonardo's polynomial sequence, for* $n \in \mathbf{N}$*, is given by:*

$$g(l_n(x), t) = \sum_{n=0}^{\infty} l_n(x) t^n = \frac{1 - 5t + t^2}{(1 - 2xt + t^3)}$$

Proof. Let $g(l_n(x), t)$ be the generating function of Leonardo's polynomial sequence $l_n(x)$, then:

$$g(l_n(x), t) - g(l_n(x) 2xt + g(l_n(x) t^3 = l_0(x) + (l_1(x) - 2l_0(x))t + (l_2(x) - 2l_1(x))t^2,$$
$$g(l_n(x), t)(1 - 2xt + t^3) = 1 - 5t + t^2,$$
$$g(l_n(x), t) = \frac{1 - 5t + t^2}{(1 - 2xt + t^3)}.$$

□

3 Leonardo's bivariate polynomials

In this section, Leonardo's bivariate polynomials will be introduced. The first terms of this sequence are given in Table 2.

Definition 3.1. *Leonardo's bivariate polynomials,* $l_n(x, y)$*, for* $n \geq 3$ *are given by:*

$$l_n(x, y) = 2x l_{n-1}(x, y) - y l_{n-3}(x, y),$$

with $l_0(x, y) = l_1(x, y) = 1$ *and* $l_2(x, y) = 3$*.*

n	$l_n(x,y)$
0	1
1	1
2	3
3	$6x-y$
4	$12x^2-2xy-y$
5	$24x^3-4x^2y-2xy-3y$
6	$48x^4-8x^3y-4x^2y-12xy+y^2$
$\vdots$	$\vdots$

Table 2. First terms of Leonardo's bivariate polynomial sequence

It is observed that with the values $x=y=1$, we have Leonardo's original sequence, as shown in Table 3.

n	$l_n(1,1)$
0	1
1	1
2	3
3	5
4	9
5	15
6	25
$\vdots$	$\vdots$

Table 3. First terms of Leonardo's bivariate polynomial sequence

Theorem 3.2. *The matrix form of Leonardo's bivariate polynomials, for $n \geq 2$ and with $n \in \mathbf{N}$, is given by:*

$$\begin{bmatrix} 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2x & 1 & 0 \\ 0 & 0 & 1 \\ -y & 0 & 0 \end{bmatrix}^n = \begin{bmatrix} l_{n+2}(x,y) & l_{n+1}(x,y) & l_n(x,y) \end{bmatrix}.$$

Proof. Analogously to the proof of Theorem 2.2, the present theorem can be validated. □

The characteristic equation of Leonardo's bivariate polynomial sequence is given by $q^3-2xq^2+y=0$, on what x and y are the polynomial variables. So, we have q_1, q_2 and q_3.

Theorem 3.3. *Binet's formula of Leonardo's bivariate polynomials, with $n \in \mathbf{Z}$, is given by:*

$$l_n(x,y) = \alpha q_1^n + \beta q_2^n + \gamma q_3^n,$$

on what q_1, q_2, q_3 are the roots of the characteristic equation $q^3-2xq^2+y=0$ and

$$\alpha = \frac{3+(-q_2-q_3)+q_2q_3}{q_1^2-q_1q_2-q_1q_3+q_2q_3}, \quad \beta = \frac{3+(-q_1-q_3)+q_1q_3}{q_2^2-q_2q_3-q_1q_2+q_1q_3}, \quad \gamma = \frac{3+(-q_1-q_2)+q_1q_2}{q_3^2+q_1q_2-q_1q_3-q_2q_3}.$$

Proof. Through the Binet formula $l_n(x, y) = \alpha q_1^n + \beta q_2^n + \gamma q_3^n$ and the recurrence of Leonardo's bivariate polynomials $l_n(x, y) = 2xl_{n-1}(x, y) - y_{n-3}(x, y)$, with the initial values $l_0(x, y) = l_1(x, y) = 1$ and $l_2(x, y) = 3$, it is possible to obtain the following system of equations:

$$\begin{cases} \alpha + \beta + \gamma & = 1 \\ \alpha q_1 + \beta q_2 + \gamma q_3 & = 1 \\ \alpha q_1^2 + \beta q_2^2 + \gamma q_3^2 & = 3 \end{cases}$$

Solving the system, we have that:

$$\begin{aligned} \alpha &= \frac{3 + (-q_2 - q_3) + q_2 q_3}{q_1^2 - q_1 q_2 - q_1 q_3 + q_2 q_3}, \\ \beta &= \frac{3 + (-q_1 - q_3) + q_1 q_3}{q_2^2 - q_2 q_3 - q_1 q_2 + q_1 q_3}, \\ \gamma &= \frac{3 + (-q_1 - q_2) + q_1 q_2}{q_3^2 + q_1 q_2 - q_1 q_3 - q_2 q_3}. \end{aligned}$$

□

Theorem 3.4. *The generator function of Leonardo's bivariate polynomial sequence, for $n \in \mathbf{N}$, is given by:*

$$g(l_n(x, y), t) = \sum_{n=0}^{\infty} l_n(x, y) t^n = \frac{1 - 5t + t^2}{(1 - 2xt + yt^3)}.$$

Proof. Be $g(l_n(x, y), t)$ the generating function of Leonardo's polynomial sequence $l_n(x, y)$, then:

$$\begin{aligned} g(l_n(x, y), t) - g(l_n(x, y)2xt + g(l_n(x, y)yt^3 &= l_0(x, y) + (l_1(x, y) - 2l_0(x, y))t \\ &\quad + (l_2(x, y) - 2l_1(x, y))t^2 \\ g(l_n(x, y), t)(1 - 2xt + yt^3) &= 1 - 5t + t^2 \\ g(l_n(x, y), t) &= \frac{1 - 5t + t^2}{(1 - 2xt + yt^3)}. \end{aligned}$$

□

4 Leonardo's complex bivariate polynomials

In this section, Leonardo's complex bivariate polynomials will be introduced.

Definition 4.1. *Leonardo's complex bivariate polynomials, $l_n(ix, y)$, for $n \geq 3$ are given by:*

$$l_n(ix, y) = 2xil_{n-1}(ix, y) - yl_{n-3}(ix, y),$$

with $l_0(ix, y) = l_1(ix, y) = 1$, $l_2(ix, y) = 3$ and $i^2 = -1$.

The first terms of this sequence are given in the following Table 4.

n	$l_n(ix,y)$
0	1
1	1
2	3
3	$6xi-y$
4	$-12x^2-2xyi-y$
5	$-24x^3i-2xyi+4x^2y-3y$
6	$48x^4+4x^2y+y^2-12xyi+8x^3yi$
$\vdots$	$\vdots$

Table 4. First terms of Leonardo's complex bivariate polynomial sequence

Theorem 4.2. *The matrix form of Leonardo's complex bivariate polynomials, for* $n \geq 2$ *and with* $n \in \mathbf{N}$*, is given by:*

$$\begin{bmatrix} 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2xi & 1 & 0 \\ 0 & 0 & 1 \\ -y & 0 & 0 \end{bmatrix}^n = \begin{bmatrix} l_{n+2}(ix,y) & l_{n+1}(ix,y) & l_n(ix,y) \end{bmatrix}.$$

Proof. Analogously to the proof of Theorem 2.2, the present theorem can be validated. □

The characteristic equation of Leonardo's complex bivariate polynomial sequence is given by $v^3-2xiv^2+y=0$, on what x and y are the polynomial variables. So, we have to v_1, v_2 and v_3 are the roots of the characteristic equation.

Theorem 4.3. *Binet's formula of Leonardo's complex bivariate polynomials, with* $n \in \mathbf{Z}$*, is given by:*

$$l_n(ix,y) = \alpha v_1^n + \beta v_2^n + \gamma v_3^n,$$

on what v_1, v_2, q_3 *are the roots of the characteristic equation* $v^3-2xiv^2+v=0$ *and*

$$\alpha = \frac{3+(-v_2-v_3)+v_2v_3}{v_1^2-v_1v_2-v_1v_3+v_2v_3}, \quad \beta = \frac{3+(-v_1-v_3)+v_1v_3}{v_2^2-v_2v_3-v_1v_2+v_1v_3}, \quad \gamma = \frac{3+(-v_1-v_2)+v_1v_2}{v_3^2+v_1v_2-v_1v_3-v_2v_3}.$$

Proof. Through the Binet formula $l_n(ix,y) = \alpha v_1^n + \beta v_2^n + \gamma v_3^n$ and the recurrence of Leonardo's complex bivariate polynomials $l_n(ix,y) = 2xil_{n-1}(ix,y) - y_{n-3}(ix,y)$, with the initial values $l_0(ix,y) = l_1(ix,y) = 1$ and $l_2(ix,y) = 3$, it is possible to obtain the following system of equations:

$$\begin{cases} \alpha+\beta+\gamma & = 1 \\ \alpha v_1+\beta v_2+\gamma v_3 & = 1 \\ \alpha v_1^2+\beta v_2^2+\gamma v_3^2 & = 3 \end{cases}.$$

Solving the system, we have that:

$$\begin{aligned} \alpha &= \frac{3+(-v_2-v_3)+v_2v_3}{v_1^2-v_1v_2-v_1v_3+v_2v_3}, \\ \beta &= \frac{3+(-v_1-v_3)+v_1v_3}{v_2^2-v_2v_3-v_1v_2+v_1v_3}, \\ \gamma &= \frac{3+(-v_1-v_2)+v_1v_2}{v_3^2+v_1v_2-v_1v_3-v_2v_3}. \end{aligned}$$

□

Theorem 4.4. *The generating function of Leonardo's complex bivariate polynomial sequence, for* $n \in \mathbf{N}$, *is given by:*

$$g(l_n(ix,y),t) = \sum_{n=0}^{\infty} l_n(ix,y)t^n = \frac{1-5t+t^2}{(1-2ixt+yt^3)}.$$

Proof. Let $g(l_n(ix,y),t)$ be the generating function of Leonardo's complex bivariate polynomial sequence $l_n(ix,y)$, then:

$$\begin{aligned} g(l_n(ix,y),t) - g(l_n(ix,y)2ixt + g(l_n(ix,y)yt^3 &= l_0(ix,y) + (l_1(ix,y) - 2l_0(ix,y))t \\ &\quad + (l_2(ix,y) - 2l_1(ix,y))t^2, \\ g(l_n(ix,y),t)(1-2ixt+yt^3) &= 1-5t+t^2, \\ g(l_n(ix,y),t) &= \frac{1-5t+t^2}{(1-2ixt+yt^3)}. \end{aligned}$$

□

5 Conclusion

This work presents a study around the Leonardo sequence, continuing the mathematical evolutionary process of this sequence, we present its polynomial form, its bivariate polynomial form and its complex bivariate polynomial form. Leonardo's sequence numbers were worked on functions of variables and explored in its complex form after the insertion of the imaginary component i. It was possible to present the recurrence of these numbers, their generating matrix, characteristic equations, Binet formula and generating function.

For future work, investigations on these polynomial numbers, bivariate polynomials and complex bivariate polynomials are proposed, finding applicability of this mathematical content in other areas.

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