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Partitions with k sizes from a set

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Abstract: Let n and k be two positive integers and let A be a set of positive integers. We define $t_A(n, k)$ to be the number of partitions of n with exactly k sizes and parts in A. As an implication of a variant of Newton's product-sum identities we present a generating function for $t_A(n, k)$. Subsequently, we obtain a recurrence relation for $t_A(n, k)$ and a divisor-sum expression for $t_A(n, 2)$. Also, we present a bijective proof for the latter expression.
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1 Introduction and statement of results

Let n be a positive integer. By a partition of n we mean a finite unordered sequence of positive integers, say $\pi = (a_1, a_2, \ldots, a_m)$, such that $a_1 + a_2 + \cdots + a_m = n$. The a_is are called the parts of the partition π, and each element in the (multi) set $\{a_1, a_2, \ldots, a_m\}$ is called a size of π.

The number of partitions of n with parts restricted to a set of positive integers was considered by Sylvester. The generating function of this enumeration and various modes of evaluating its exact expressions were presented in a classical text book by J. Riordan [5].

In this note we are concerned with the partitions which satisfies the following conditions:

1. parts restricted to a set of positive integers,
2. the number of sizes is restricted to k, a positive integer.

The formal definition of its enumeration is given below.

Definition 1.1. *Let n and k be two positive integers. Let A be a set of positive integers. We define $t_A(n,k)$ to be the number of partitions of n with exactly k sizes and parts from the set A.*

The function $t_{\mathbb{N}}(n,k)$ has been investigated by many authors [2–4]. In this note, we obtain a generating function for $t_A(n,k)$. The sum of product form of this generating function allows us to apply a variant of Newton's product-sum identity in order to obtain a recurrence relation for $t_A(n,k)$. We then confine to the case $k=2$ and obtain a divisor-sum expression for $t_A(n,2)$.

Following is the variant of Newton's product-sum identities which plays a vital role in obtaining a recurrence relation for $t_A(n,k)$.

Lemma 1.2. *Let $\sum_{n=1}^{\infty} c_n$ be a converging series of real numbers with $c_n \geq 0$. Define*

$$S_0 = 1,$$

$$S_k = \sum_{\{i_1,\ldots,i_k\}\subset\mathbb{N}} c_{i_1}c_{i_2}\cdots c_{i_k}$$

and

$$P_k = \sum_{j=1}^{\infty} c_j^k$$

for every $k \in \mathbb{N}$. If $|P_k| < \infty$ for every $k \in \mathbb{N}$, then we have

$$(-1)^{k+1}(k+1)S_{k+1} = -(P_{k+1}S_0 - P_kS_1 + \cdots + (-1)^k P_1 S_k). \tag{1}$$

Now we list the main results of this note. Proof of these results are provided in the next section.

Theorem 1.3. *Let k be a positive integer and let A be a set of positive integers. We have*

$$\sum_{n=1}^{\infty} t_A(n,k)x^k = \sum_{\{a_{i_1},\ldots,a_{i_k}\}\subseteq A} \frac{x^{a_{i_1}}}{1-x^{a_{i_1}}}\cdots\frac{x^{a_{i_k}}}{1-x^{a_{i_k}}}. \tag{2}$$

Using this generating function we will get the following recurrence relation for $t_A(n,k)$.

Theorem 1.4. *Let k be a positive integer and let A be a set of positive integers. We have*

$$(-1)^{k+1}(k+1)t_A(n,k+1) = -\tau_A(n,k+1) + \sum_{r=1}^{k}(-1)^{r-1}\tau_A(n,k+1-r) * t_A(n,r), \tag{3}$$

where $*$ denotes the convolution operator given by

$$(g*f)(n) = \sum_{k=1}^{n-1} g(k)f(n-k)$$

and $\tau_A(n,k)$ is defined by

$$\tau_A(n,k) = \sum_{\substack{d\mid n\\ d\geq k}} \binom{d-1}{k-1}\chi_A\left(\frac{n}{d}\right),$$

where $\chi_A(n)$ denotes the characteristic function of A.

As an immediate consequence of Theorem 1.4, we have the following result.

Corollary 1.5 (G. E. Andrews [1]). *Let $n \geq 2$ be a positive integer. We have*

$$t_{\mathbb{N}}(n, 2) = \frac{\sum_{k=1}^{n-1} \tau(k)\tau(n-k) - \sigma(n) + \tau(n)}{2}, \tag{4}$$

where $\tau(n)$ (respectively, $\sigma(n)$) denotes the number of (respectively, sum of) positive divisors of n.

Definition 1.6. *Let n be positive integer and let A be a set of positive integers. We define $\tau_A(n)$ to be the number of divisors of n such that each divisor is from the set A. In notation,*

$$\tau_A(n) = \sum_{\substack{d|n \\ d \in A}} 1. \tag{5}$$

Next we have a generalisation of the Corollary 1.5.

Theorem 1.7. *Let n be a positive integer and let A be a set of positive integers. Then we have*

$$t_A(n, 2) = \frac{\sum_{k=1}^{n-1} \tau_A(k)\tau_A(n-k) - \sum_{\substack{d|n \\ d \in A}} \frac{n}{d} + \tau_A(n)}{2}. \tag{6}$$

2 Generating function and a recurrence for $t_A(n, k)$

For the sake of completeness, we present a proof for Lemma 1.2. Our proof is a simple variant of D. Zeilberger's combinatorial proof [6].

2.1 Proof of Lemma 1.2

Fix $k + 1 \in \mathbb{N}$. Consider the set of ordered pairs, (A, j^l), denoted $\mathscr{A}(k + 1)$, where

(i) A is a subset of $\mathbb{N}$ with $|A| \leq k + 1$, where $|A|$ denotes the number of elements of A,

(ii) j is a member of $\mathbb{N}$,

(iii) $|A| + l = k + 1$,

(iv) $l \geq 0$ and if $l = 0$, then $j \in A$.

Define the weight of (A, j^l), denoted $w(A, j^l)$, by

$$w(A, j^l) = (-1)^{|A|} \left(\prod_{a \in A} c_a \right) c_j^l.$$

We aim at proving the following identity

$$\sum_{l=0}^{k} (-1)^l S_l P_{k+1-l} + (-1)^{k+1}(k + 1)S_{k+1} = 0. \tag{7}$$

To that end, we will show that the left-hand side sum of the equation above equals the sum of all the weights of elements of $\mathscr{A}(k + 1)$.

Case i. When $|A| = 0$, we have $l = k+1$, $j \in \mathbb{N}$ and

$$\begin{aligned} w(A, j^l) &= (-1)^0 c_j^l \\ &= c_j^{k+1}. \end{aligned}$$

While summing the term above over $j \in \mathbb{N}$, we have

$$\begin{aligned} \sum_{j=1}^{\infty} w(A, j^l) &= \sum_{j=1}^{\infty} c_j^{k+1} \\ &= P_{k+1}. \end{aligned}$$

Case ii. Assume $1 \le |A| \le k$. Then $1 \le l \le k$ and $j \in \mathbb{N}$.
Let $r = |A|$. We have

$$w(A, j^l) = (-1)^r \left(\prod_{a \in A} c_a \right) c_j^{(k+1)-r}.$$

Taking sum over the subsets $A \subset \mathbb{N}$ having $|A| = r$, we have

$$\sum_{\substack{A \subset \mathbb{N} \\ |A| = r}} w(A, j^l) = (-1)^r S_r c_j^{k+1-r}.$$

Now taking sum over $j \in \mathbb{N}$, we have

$$\begin{aligned} \sum_{j \in \mathbb{N}} \sum_{\substack{A \subset \mathbb{N} \\ |A| = r}} w(A, j^l) &= (-1)^r S_r \sum_{j \in \mathbb{N}} c_j^{(k+1)-r} \\ &= (-1)^r S_r P_{(k+1)-r}. \end{aligned}$$

Case iii. Assume $|A| = k+1$. Then $l = 0$ and $j \in A$. We have

$$\begin{aligned} w(A, j^l) &= (-1)^{k+1} \left(\prod_{a \in A} C_a \right) c_j^0 \\ &= (-1)^{k+1} \left(\prod_{a \in A} C_a \right). \end{aligned}$$

Taking summation over $j \in A$, we have

$$\sum_{j \in A} w(A, j^l) = (-1)^{k+1} (k+1) \left(\prod_{a \in A} c_a \right).$$

Now taking summation over A having $|A| = k+1$, we have

$$\begin{aligned} \sum_{j \in \mathbb{A}} \sum_{\substack{A \subset \mathbb{N} \\ |A| = k+1}} w(A, j^l) &= (-1)^{k+1} (k+1) \sum_{\substack{A \subset \mathbb{N} \\ |A| = k+1}} \left(\prod_{a \in A} c_a \right) \\ &= (-1)^{k+1} (k+1) S_{k+1}. \end{aligned}$$

This completes the claim.

Define $T : \mathscr{A}(k+1) \to \mathscr{A}(k+1)$ by

$$T(A, j^l) = \begin{cases} (A \setminus \{j\}, j^{l+1}), & j \in A, \\ (A \cup \{j\}, j^{l-1}), & j \notin A. \end{cases}$$

One can verify the following properties:

(a) $w(T((A, j^l))) = -w((A, j^l))$,

(b) $T(T((A, j^l))) = (A, j^l)$.

The above two observations lead to the conclusion that, if we take the sum over the terms

$$w(T((A, j^l))) + w(A, j^l) = 0,$$

then we will get the sum on the left-hand side of (7) as zero. Now the proof is completed. $\square$

2.2 Proof of Theorem 1.3

Fix $k \in \mathbb{N}$. Let $\{a_{i_1}, \ldots, a_{i_k}\}$ be a subset of A. Define

$$R_{\{a_{i_1},\ldots,a_{i_k}\}}(n) = \#\{\text{partitions of } n \text{ having exactly } k \text{ sizes, namely, } a_{i_1}, \ldots, a_{i_k}\}$$

and

$$p_{\{a_{i_1},\ldots,a_{i_k}\}}(n) = \#\{\text{partitions of } n \text{ having parts from the set } \{a_{i_1}, \ldots, a_{i_k}\}\}.$$

We recall the following well-known generating function of $p_{\{a_{i_1},\ldots,a_{i_k}\}}(n)$:

$$\sum_{n=0}^{\infty} p_{\{a_{i_1},\ldots,a_{i_k}\}}(n)x^n = \frac{1}{1 - x^{a_{i_1}}} \cdots \frac{1}{1 - x^{a_{i_k}}}.$$

Let $(b_1, \ldots, b_s)$ be a partition of n with parts from the set $\{a_{i_1}, \ldots, a_{i_k}\}$. Then the mapping

$$(b_1, \ldots, b_s) \to (b_1, \ldots, b_s, a_{i_1}, \ldots, a_{i_k})$$

establishes a one-to-one correspondence between the following sets:

(i) the set of all partitions of n with parts from the set $\{a_{i_1}, \ldots, a_{i_k}\}$

(ii) the set of all partitions of $n + a_{i_1} + \cdots + a_{i_k}$ having k sizes, namely, $a_{i_1}, \ldots, a_{i_k}$.

That is

$$R_{\{a_{i_1},\ldots,a_{i_k}\}}(n + a_{i_1} + \cdots + a_{i_k}) = p_{\{a_{i_1},\ldots,a_{i_k}\}}(n).$$

Whence, we have

$$\begin{aligned} \sum_{m \geq a_{i_1} + \cdots + a_{i_k}} R_{\{a_{i_1},\ldots,a_{i_k}\}}(m)x^m &= \sum_{n=0}^{\infty} R_{\{a_{i_1},\ldots,a_{i_k}\}}(n + a_{i_1} + \cdots + a_{i_k})x^{n+a_{i_1}+\cdots+a_{i_k}} \\ &= \sum_{n=0}^{\infty} p_{\{a_{i_1},\ldots,a_{i_k}\}}(n)x^{n+a_{i_1}+\cdots+a_{i_k}} \\ &= x^{a_{i_1}+\cdots+a_{i_k}} \sum_{n=0}^{\infty} p_{\{a_{i_1},\ldots,a_{i_k}\}}(n)x^n \\ &= \frac{x^{a_{i_1}}}{1 - x^{a_{i_1}}} \cdots \frac{x^{a_{i_k}}}{1 - x^{a_{i_k}}}. \end{aligned}$$

Since

$$t_A(n,k) = \sum_{\substack{\{a_{i_1},\dots,a_{i_k}\}\subseteq A \\ n\geq a_{i_1}+\cdots+a_{i_k}}} R_{\{a_{i_1},\dots,a_{i_k}\}}(n),$$

we have

$$\sum_{n=1}^{\infty} t_A(n,k)x^n = \sum_{\substack{\{a_{i_1},\dots,a_{i_k}\}\subseteq A \\ n\geq a_{i_1}+\cdots+a_{i_k}}} R_{\{a_{i_1},\dots,a_{i_k}\}}(n)x^n \tag{8}$$

$$= \sum_{\{a_{i_1},\dots,a_{i_k}\}\subseteq A} \frac{x^{a_{i_1}}}{1-x^{a_{i_1}}}\cdots\frac{x^{a_{i_k}}}{1-x^{a_{i_k}}}, \tag{9}$$

which is the desired end. □

2.3 Proof of Theorem 1.4

The above sum-of-product form of generating function is suitable for applying Lemma 1.2 in deriving a recurrence relation for the function $t_A(n,k)$.

Define

$$c_n(x) = \begin{cases} \dfrac{x^n}{1-x^n}, & \text{if } n \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Define $S_0 = 1$. For $r \geq 1$, define

$$S_r(x) = \sum_{\{j_1,\dots,j_r\}\subset\mathbb{N}} c_{j_1}(x)\cdots c_{j_r}(x).$$

Then we have, from Theorem 1.3, that

$$S_r(x) = \sum_{\{a_{i_1},\dots,a_{i_r}\}\subseteq A} \frac{x^{a_{i_1}}}{1-x^{a_{i_1}}}\cdots\frac{x^{a_{i_r}}}{1-x^{a_{i_r}}}$$
$$= \sum_{n=1}^{\infty} t_A(n,r)x^n.$$

Define

$$P_r(x) = \sum_{n=1}^{\infty} c_n(x)^r.$$

Then we have

$$P_r(x) = \sum_{a\in A}\left(\frac{x^a}{1-x^a}\right)^r$$
$$= \sum_{a\in A}\frac{x^{ra}}{(1-x^a)^r}$$
$$= \sum_{a\in A}\left(x^{ra} + \binom{r-1+1}{r-1}x^{(r+1)a} + \binom{r-1+2}{r-1}x^{(r+2)a} + \cdots\right)$$

$$= \sum_{n=1}^{\infty} \left(\sum_{\substack{d \geq r \\ d|n \\ \frac{n}{d} \in A}} \binom{(d-r)+(r-1)}{r-1} \right) x^n$$

$$= \sum_{n=1}^{\infty} \tau_A(n,r) x^n.$$

Then Lemma 1.2 gives

$$(-1)^{k+1}(k+1)S_{k+1}(x) = -P_{k+1}(x) + \sum_{r=1}^{k} (-1)^{r-1} P_{k+1-r}(x) S_r(x).$$

Now substituting the series expansion of $S_r(x)$ and $P_r(x)$ in the identity above and equating the coefficients of like powers of x on both sides gives the recurrence relation of Theorem 1.4. $\square$

3 Divisor-sum expression for $t_A(n, 2)$

In this section we provide two proofs for Theorem 1.7. First proof is based on Theorem 1.4, and the second proof is based upon a correspondence between the partitions of n having exactly two distinct sizes and the divisors of $n - ta$ which are different from a, with a varying over A and $t \in \mathbb{N}$ such that $n - ta \geq 1$.

3.1 Proof of Theorem 1.7 using Theorem 1.4

If we fix $k = 2$ then Theorem 1.4 gives

$$\begin{aligned} 2t_A(n,2) &= -\sum_{d|n} (d-1) \chi_A\left(\frac{n}{d}\right) + \sum_{k=1}^{n-1} \tau_A(k)\tau_A(n-k) \\ &= -\sum_{\frac{n}{d}|n} \left(\frac{n}{d} - 1\right) \chi_A(d) + \sum_{k=1}^{n-1} \tau_A(k)\tau_A(n-k) \\ &= -\sum_{\substack{d \in A \\ d|n}} \left(\frac{n}{d} - 1\right) + \sum_{k=1}^{n-1} \tau_A(k)\tau_A(n-k) \\ &= -\sum_{\substack{d \in A \\ d|n}} \frac{n}{d} + \tau_A(n) + \sum_{k=1}^{n-1} \tau_A(k)\tau_A(n-k), \end{aligned}$$

which is the desired end. $\square$

3.2 A bijective proof of Theorem 1.7

If there is a partition of n with two distinct sizes, say a and b, each from a set of positive integers, say A. Then we can write $n = ta + sb$, where $a, b \in A$ with $a \neq b$ and $t, s \in \mathbb{N}$.

The equality above can be written as

$$n - ta = sb. \tag{10}$$

We observe that for a fixed $a \in A$, we have b as a divisor of $n - ta$ with $b \in A$ and $b \neq a$. While varying $t \in \mathbb{N}$ with $n - ta \geq 1$, we get the number of such divisors as

$$\sum_{\substack{t\in\mathbb{N}\\ n-ta\geq 1}} \tau_{A-\{a\}}(n - ta).$$

Also, we observe that corresponding to each such divisor there is a partition of n with two distinct sizes, namely, a and b.

Now, varying a over A and summing the expression above, we see that the terms ta and sb in (10) will get commuted, and as the consequence we get

$$2t_A(n, 2) = \sum_{\substack{t\in\mathbb{N}\\ a\in A\\ n-ta\geq 1}} \tau_{A-\{a\}}(n - ta). \tag{11}$$

Now we observe that

$$\tau_{A-\{a\}}(n - ta) = \begin{cases} \tau_A(n - ta) \text{ if } a \nmid n; \\ \tau_A(n - ta) - 1 \text{ if } a \mid n. \end{cases}$$

This gives

$$\sum_{\substack{t\in\mathbb{N}\\ a\in A\\ n-ta\geq 1}} \tau_{A-\{a\}}(n - ta) = \sum_{\substack{t\in\mathbb{N}\\ a\in A\\ n-ta\geq 1}} \tau_A(n - ta) - \sum_{\substack{a\in A\\ a\mid n}} \lfloor \frac{n-1}{a} \rfloor. \tag{12}$$

Fix $k \in \{1, 2, \ldots, n - 1\}$. Consider the equality $n - ta = k$, where $t \in \mathbb{N}$ and $a \in A$. Then the number of pairs, say (t, a), satisfying this equality equals $\tau_A(n - k)$. In notation

$$\sum_{\substack{t\in\mathbb{N}\\ a\in A\\ n-ta=k}} 1 = \tau_A(n - k).$$

This gives

$$\begin{aligned} \sum_{\substack{t\in\mathbb{N}\\ a\in A\\ n-ta=k}} \tau_A(n - ta) &= \sum_{\substack{t\in\mathbb{N}\\ a\in A\\ n-ta=k}} \tau_A(k) \\ &= \tau_A(n - k)\tau_A(k). \end{aligned}$$

Consequently,

$$\sum_{\substack{t\in\mathbb{N}\\ a\in A\\ n-ta\geq 1}} \tau_A(n - ta) = \sum_{k=1}^{n-1} \tau_A(k)\tau_A(n - k). \tag{13}$$

Also we observe that, if $a \mid n$ and $a \in A$, then we have $\lfloor \frac{n-1}{a} \rfloor = \lfloor \frac{n}{a} - \frac{1}{a} \rfloor = \frac{n}{a} - 1$. This gives the relation

$$\sum_{\substack{a\in A\\ a\mid n}} \lfloor \frac{n-1}{a} \rfloor = \sum_{\substack{a\in A\\ a\mid n}} \frac{n}{a} - \tau_A(n). \tag{14}$$

If we substitute the right-hand side expression of (12) with values from (13) and (14), then (11) gives the desired expression for $t_A(n, 2)$. $\square$

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