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Sums involving generalized harmonic and Daehee numbers

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Abstract: In this paper, we establish some sums involving generalized harmonic and Daehee numbers which are derived from the generating functions. For example, for $n, r \geq 1$,

$$\sum_{i=0}^{n} H\left(i, r-1, \alpha\right) H_{n-i}^{r}\left(\alpha\right) = \sum_{l_1+l_2+\cdots+l_{r+1}=n} H_{l_1}(\alpha) H_{l_2}(\alpha) \cdots H_{l_{r+1}}(\alpha).$$

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1 Introduction

The harmonic numbers are defined by

$$H_0 = 0 \text{ and } H_n = \sum_{i=1}^{n} \frac{1}{i} \text{ for } n \geq 1.$$

It is well known that

$$H_n = \int_0^1 \frac{1-t^n}{1-t} dt = \gamma + \psi\left(n+1\right),$$

where γ denotes the Euler–Mascheroni constant, defined by

$$\gamma = \lim_{n \longrightarrow \infty} \left(\sum_{i=1}^{n} \frac{1}{i} - \log n \right) = -\psi(1) \approx 0.577215664901532860606512\ldots$$

Harmonic numbers are closely related the Riemann ξ-function defined by

$$\xi(s) = \sum_{i=1}^{\infty} \frac{1}{i^s} = \prod_{p} \left(1 - p^{-s}\right)^{-1},$$

where the product is over all primes p.

These numbers have been generalized by some authors [1,2,4,9,16,18].

In [9], for any $\alpha \in \mathbb{R}^+$, the generalized harmonic numbers $H_n(\alpha)$ are defined by

$$H_0(\alpha) = 0 \text{ and } H_n(\alpha) = \sum_{i=1}^{n} \frac{1}{i\alpha^i} \text{ for } n \geq 1.$$

For $\alpha = 1, H_n(1) = H_n$ are the usual harmonic numbers and the generating function of the generalized harmonic numbers is

$$\sum_{n=1}^{\infty} H_n(\alpha) x^n = -\frac{\ln\left(1 - \frac{x}{\alpha}\right)}{1 - x}.$$

In [15], for the generalized harmonic numbers $H_n(\alpha)$, the authors defined the generalized hyperharmonic numbers of order r, $H_n^r(\alpha)$ as follows:

Definition 1. *For $r < 0$ or $n \leq 0$, $H_n^r(\alpha) = 0$ and for $n \geq 1$, the generalized hyperharmonic numbers of order r, $H_n^r(\alpha)$, are defined by*

$$H_n^r(\alpha) = \sum_{i=1}^{n} H_i^{r-1}(\alpha), \quad r \geq 1,$$

where $H_n^0(\alpha) = \frac{1}{n\alpha^n}$.

For $\alpha = 1$, $H_n^r(1) = H_n^r$ are the hyperharmonic numbers of order r. The generating function of the generalized hyperharmonic numbers of order r is

$$\sum_{n=1}^{\infty} H_n^r(\alpha) x^n = -\frac{\ln\left(1 - \frac{x}{\alpha}\right)}{(1 - x)^r}. \tag{1}$$

In [4, 18], the generalized harmonic numbers $H(n, r)$ of rank r are defined as for $n \geq 1$ and $r \geq 0$,

$$H(n, r) = \sum_{1 \leq n_0 + n_1 + \cdots + n_r \leq n} \frac{1}{n_0 n_1 \cdots n_r}.$$

It is clear that $H(n, 0) = H_n$. The generating function of the generalized harmonic numbers $H(n, r)$ of rank r is defined by

$$\sum_{n=0}^{\infty} H(n, r) x^n = \frac{\left(-\ln(1 - x)\right)^{r+1}}{1 - x}.$$

In [8], inspired from works [4, 15, 18], $H(n, r, \alpha)$ are defined as for $n \geq 1$ and $r \geq 0$,

$$H(n, r, \alpha) = \sum_{1 \leq n_0 + n_1 + \cdots + n_r \leq n} \frac{1}{n_0 n_1 \cdots n_r \alpha^{n_0 + n_1 + \cdots + n_r}}.$$

For $\alpha = 1$, $H(n,r,1) = H(n,r)$. The generating function of the generalized harmonic numbers of rank r, $H(n,r,\alpha)$, is given by

$$\sum_{n=0}^{\infty} H(n,r,\alpha) x^n = \frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}}{1-x}. \tag{2}$$

The Cauchy numbers of order r, C_n^r, are defined by the generating functions to be

$$\left(\frac{x}{\ln(1+x)}\right)^r = \sum_{n=0}^{\infty} C_n^r \frac{x^n}{n!}. \tag{3}$$

The Daehee numbers of order r, D_n^r, are defined by the generating functions to be

$$\left(\frac{\ln(1+x)}{x}\right)^r = \sum_{n=0}^{\infty} D_n^r \frac{x^n}{n!}. \tag{4}$$

For $r = 1$, $D_n^1 = D_n$ are Daehee numbers. It is clear that

$$D_0 = 1, D_1 = -\frac{1}{2}, \ldots, D_n = (-1)^n \frac{n!}{n+1}.$$

The derangement numbers d_n are defined by the generating functions to be

$$\frac{e^{-x}}{1-x} = \sum_{n=0}^{\infty} d_n \frac{x^n}{n!} \tag{5}$$

and $d_n = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}$ [5].

The generalized geometric series are given by for $a, b \in \mathbb{Z}^+$,

$$\sum_{n=b}^{\infty} \binom{a+n-b}{n-b} x^n = \frac{x^b}{(1-x)^{a+1}}, \tag{6}$$

and the exponential generating function is

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$ and $G(x) = \sum_{n=0}^{\infty} b_n x^n$ be two generating functions. The product of these functions is given as follows:

$$F(x)G(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n, \tag{7}$$

where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

Recently, there are many studies including generalized harmonic and special numbers which are obtained by generating functions [6–8, 10–14, 20].

In [17], Rim et al. investigated some identities with hyperharmonic, Daehee and derangement numbers. For example, for any positive integer n,

$$\sum_{i=0}^{n} H_i^r \frac{(-1)^{n-i}}{(n-i)!} = \sum_{i=0}^{n} H_i^{r-1} \frac{d_{n-i}}{(n-i)!}.$$

In [8], Duran et al. obtained sums including generalized harmonic numbers and special numbers. For example, for any positive integers n, r and m,

$$H(n,r,\alpha)=\sum_{i=0}^{n}\sum_{j=0}^{i}(-1)^{n-j-r}\binom{m-1}{n-i}\frac{H_j^m(\alpha)\,s(i-j,r)r!}{\alpha^{i-j}(i-j)!},$$

where the Stirling numbers of the first kind $s(n,i)$ are given by

$$x^{\underline{n}}=\sum_{i=0}^{n}s(n,i)x^i,$$

where for $n\geq 0$, $s(n,0)=\delta_{n0}$, δ_{ni} is the Kronecker delta [3, 19]. $x^{\underline{n}}$ stands for the falling factorial defined by $x^{\underline{n}}=x(x-1)\ldots(x-n+1)$.

In [11], Kim et al. gave some new identities involving harmonic and hyperharmonic numbers which are derived from the transfer formula for the associated sequences. For example, for $n,r\geq 1$ and $1\leq k\leq n$,

$$\begin{aligned}&\binom{(r+3)n-k-1}{n-k}(n-1)^{\underline{n-k}}\\ =\;&\sum_{a=k}^{n}\sum_{l=0}^{n-a}\sum_{i=0}^{n}\sum_{j_1+j_2+\cdots+j_n=a-k}\left(\sum_{m_1=1}^{j_1+1}\cdots\sum_{m_n=1}^{j_n+1}m_1\cdots m_nH_{m_1}^r\cdots H_{m_n}^r\right)l!r^l\\ &\times\binom{n+l-1}{l}\binom{n-1}{a-1}s(n-a,l)(a-1)^{\underline{n-k}}.\end{aligned}$$

2 Sums with the generalized harmonic numbers of rank r and special numbers

This section, we will give some sums involving these numbers, using the generating functions of the generalized harmonic numbers of rank r and special numbers.

Theorem 2.1. *Let n be a positive integer. For $r\geq 1$,*

$$\begin{aligned}\sum_{i=0}^{n}H(i,r-1,\alpha)H_{n-i}^r(\alpha)&=\sum_{l_1+l_2+\cdots+l_{r+1}=n}H_{l_1}(\alpha)H_{l_2}(\alpha)\cdots H_{l_{r+1}}(\alpha)\\ &=\sum_{i=0}^{n}(-1)^i\binom{n-i-1}{r}\frac{D_i^{r+1}}{i!\alpha^{i+r+1}}.\end{aligned}$$

Proof. By (1) and (2), we consider that

$$\frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^r}{1-x}.\frac{-\ln\left(1-\frac{x}{\alpha}\right)}{(1-x)^r}=\left(\sum_{n=0}^{\infty}H(n,r-1,\alpha)x^n\right)\left(\sum_{n=0}^{\infty}H_n^r(\alpha)x^n\right),$$

and using (7), equals

$$\sum_{n=0}^{\infty}\sum_{i=0}^{n}H(i,r-1,\alpha)H_{n-i}^r(\alpha)x^n, \tag{8}$$

and

$$
\begin{aligned}
\frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}}{(1-x)^{r+1}} &= \left(\frac{-\ln\left(1-\frac{x}{\alpha}\right)}{1-x}\right)\times\left(\frac{-\ln\left(1-\frac{x}{\alpha}\right)}{1-x}\right)\times\cdots\times\left(\frac{-\ln\left(1-\frac{x}{\alpha}\right)}{1-x}\right)\\
&= \left(\sum_{l_1=0}^{\infty}H_{l_1}(\alpha)x^{l_1}\right)\left(\sum_{l_2=0}^{\infty}H_{l_2}(\alpha)x^{l_2}\right)\dots\left(\sum_{l_{r+1}=0}^{\infty}H_{l_{r+1}}(\alpha)x^{l_{r+1}}\right)\\
&= \sum_{n=0}^{\infty}\sum_{l_1+l_2+\cdots+l_{r+1}=n}H_{l_1}(\alpha)H_{l_2}(\alpha)\cdots H_{l_{r+1}}(\alpha)x^n. \qquad (9)
\end{aligned}
$$

Also, from (4) and (6), we have

$$
\begin{aligned}
\frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}}{(1-x)^{r+1}} &= \left(\frac{\ln\left(1-\frac{x}{\alpha}\right)}{-x}\right)^{r+1}\frac{x^{r+1}}{(1-x)^{r+1}}\\
&= \frac{1}{\alpha^{r+1}}\sum_{n=0}^{\infty}(-1)^n D_n^{r+1}\frac{x^n}{n!\alpha^n}\sum_{n=r}^{\infty}\binom{n}{r}x^{n+1}\\
&= \sum_{n=0}^{\infty}(-1)^n D_n^{r+1}\frac{x^n}{n!\alpha^{n+r+1}}\sum_{n=r+1}^{\infty}\binom{n-1}{r}x^n\\
&= \sum_{n=0}^{\infty}\sum_{i=0}^{n}(-1)^i\binom{n-i-1}{r}\frac{D_i^{r+1}}{i!\alpha^{i+r+1}}x^n. \qquad (10)
\end{aligned}
$$

Hence from (8), (9) and (10), the desired results are obtained. □

Theorem 2.2. *Let n and r be positive integers. For $m \geq 2$,*

$$
\sum_{i=0}^{n}H\left(i,m-2,\alpha\right)H_{n-i}^{rm-1}\left(\alpha\right)=\sum_{l_1+l_2+\cdots+l_m=n}H_{l_1}^r(\alpha)H_{l_2}^r(\alpha)\cdots H_{l_m}^r(\alpha),
$$

and

$$
\sum_{i=0}^{n}H\left(i,rm-2,\alpha\right)H_{n-i}^{m-1}\left(\alpha\right)=\sum_{l_1+l_2+\cdots+l_m=n}H(l_1,r-1,\alpha)H(l_2,r-1,\alpha)\cdots H(l_m,r-1,\alpha).
$$

Proof. The proof is similar to the proof of Theorem 2.1. □

Theorem 2.3. *Let n be a positive integer. For $r \geq 1$,*

$$
\sum_{i=0}^{n}(-1)^i\frac{C_i}{\alpha^{i-1}i!}H\left(n-i+1,r+1,\alpha\right)=H\left(n,r,\alpha\right).
$$

Proof. From (2) and (3), we write

$$
\begin{aligned}
\sum_{n=0}^{\infty}H\left(n,r,\alpha\right)x^n &= \frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+2}}{1-x}\frac{-x/\alpha}{\ln\left(1-\frac{x}{\alpha}\right)}\frac{\alpha}{x}\\
&= \sum_{n=0}^{\infty}H\left(n,r+1,\alpha\right)x^{n-1}\sum_{n=0}^{\infty}(-1)^n C_n\frac{x^n}{\alpha^{n-1}n!}\\
&= \sum_{n=0}^{\infty}H\left(n+1,r+1,\alpha\right)x^{n}\sum_{n=0}^{\infty}(-1)^n C_n\frac{x^n}{\alpha^{n-1}n!},
\end{aligned}
$$

and by (7)

$$\sum_{n=0}^{\infty} H(n,r,\alpha)\,x^n = \sum_{n=0}^{\infty}\sum_{i=0}^{n}(-1)^i\,\frac{C_i}{\alpha^{i-1}i!}H(n-i+1,r+1,\alpha)\,x^n$$

as claimed. So, the proof is complete. □

Theorem 2.4. *Let n be a positive integer. For $r \geq 1$,*

$$\sum_{j=0}^{n}\sum_{i=0}^{j} H(i,r-1,\alpha)\,H_{j-i}(\alpha)\,\frac{(-1)^{n-j}}{(n-j)!} \;=\; \sum_{i=0}^{n} H(i,r,\alpha)\,\frac{d_{n-i}}{(n-i)!}$$

and

$$\sum_{i=0}^{n}\frac{(-1)^i\,d_{n-i-r-1}D_i^{r+1}}{\alpha^{i+r+1}i!\,(n-i-r-1)!} \;=\; \sum_{i=0}^{n} H(i,r,\alpha)\,\frac{(-1)^{n-i}}{(n-i)!}.$$

Proof. By (7), we observe that

$$\begin{aligned}\frac{\left(-\ln(1-\frac{x}{\alpha})\right)^{r+1}}{(1-x)^2}e^{-x} &= \sum_{n=0}^{\infty} H(n,r,\alpha)\,x^n\sum_{n=0}^{\infty}\frac{d_n}{n!}x^n\\ &= \sum_{n=0}^{\infty}\sum_{i=0}^{n} H(i,r,\alpha)\,\frac{d_{n-i}}{(n-i)!}x^n \qquad (11)\end{aligned}$$

and

$$\begin{aligned}\frac{\left(-\ln(1-\frac{x}{\alpha})\right)^{r+1}}{(1-x)^2}e^{-x} &= \frac{\left(-\ln(1-\frac{x}{\alpha})\right)^{r}}{1-x}\,\frac{-\ln(1-\frac{x}{\alpha})}{1-x}e^{-x}\\ &= \sum_{n=0}^{\infty} H(n,r-1,\alpha)\,x^n\sum_{n=0}^{\infty}H_n(\alpha)\,x^n\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n\\ &= \sum_{n=0}^{\infty}\sum_{i=0}^{n} H(i,r-1,\alpha)\,H_{n-i}(\alpha)\,x^n\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n\\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{i=0}^{j} H(i,r-1,\alpha)\,H_{j-i}(\alpha)\,\frac{(-1)^{n-j}}{(n-j)!}x^n. \qquad (12)\end{aligned}$$

From here, (11) and (12) yield the desired result.
From (7), we write

$$\begin{aligned}\frac{\left(-\ln(1-\frac{x}{\alpha})\right)^{r+1}}{1-x}e^{-x} &= \sum_{n=0}^{\infty} H(n,r,\alpha)\,x^n\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n\\ &= \sum_{n=0}^{\infty}\sum_{i=0}^{n} H(i,r,\alpha)\,\frac{(-1)^{n-i}}{(n-i)!}x^n, \qquad (13)\end{aligned}$$

and, by (4) and (5),

$$\begin{aligned}
\frac{\left(-\ln(1-\frac{x}{\alpha})\right)^{r+1}}{1-x}e^{-x} &= \left(\frac{\ln(1-\frac{x}{\alpha})}{-x}\right)^{r+1}\frac{e^{-x}}{1-x}x^{r+1} \\
&= \sum_{n=0}^{\infty}(-1)^n\frac{D_n^{r+1}}{\alpha^{n+r+1}n!}x^n\sum_{n=0}^{\infty}\frac{d_n}{n!}x^{n+r+1} \\
&= \sum_{n=0}^{\infty}(-1)^n\frac{D_n^{r+1}}{\alpha^{n+r+1}n!}x^n\sum_{n=0}^{\infty}\frac{d_{n-r-1}}{(n-r-1)!}x^n \\
&= \sum_{n=0}^{\infty}\sum_{i=0}^{n}\frac{(-1)^i d_{n-i-r-1}D_i^{r+1}}{\alpha^{i+r+1}i!\,(n-i-r-1)!}x^n. \qquad (14)
\end{aligned}$$

With the help of (13) and (14), we have a relation between the generalized harmonic numbers of rank r, $H(n,r,\alpha)$, and Daehee numbers of order r. $\square$

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