Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2022, Volume 28, Number 1, Pages 64–74
DOI: 10.7546/nntdm.2022.28.1.64-74

Determinantal and permanental representations of companion sequences associated to the r-Fibonacci sequence

Hacène Belbachir[1] **and Ihab-Eddine Djellas**[2]

[1] USTHB, Faculty of Mathematics, RECITS Laboratory
Po. Box 32, El Alia, 16111, Bab Ezzouar, Algiers, Algeria
e-mails: hbelbachir@usthb.dz, hacenebelbachir@gmail.com

[2] USTHB, Faculty of Mathematics, RECITS Laboratory
Po. Box 32, El Alia, 16111, Bab Ezzouar, Algiers, Algeria
e-mails: idjellas@usthb.dz, ihebusthb@gmail.com

Received: 22 January 2021 **Revised:** 24 November 2021
Accepted: 11 February 2022 **Online First:** 14 February 2022

Abstract: Recently companion sequences of r-Fibonacci sequence were defined. The aim of this paper is to give some determinantal and permanental representations of these sequences via Hessenberg matrices. Several representations of classical sequences and polynomials are established. We conclude by using our representations to give n consecutive terms of companion sequences simultaneously.

Keywords: Generalized bivariate r-Fibonacci polynomials, Generalized bivariate r-Lucas polynomials, Companion sequences, Determinant, Permanent, Hessenberg matrix.
2020 Mathematics Subject Classification: 15A15, 11B39, 11B37.

1 Introduction

The generalized bivariate r–Fibonacci polynomials were defined in [10] for $r \geq 1$; see also [1] as follows,

$$U_n^r = xU_{n-1}^r + yU_{n-r-1}^r, \text{ for } n > r, \tag{1}$$

with x, y are variables and boundary conditions $U_0^r = 0$, $U_k^r = x^{k-1}$ for $1 \leq k \leq r$.

For any integer $1 \leq s \leq r$, Abbad et al. [1], defined the companion sequences family $(V_n^{r,s})$ related to the r-Fibonacci sequence (U_n^r) as follows,

$$V_n^{r,s} = xV_{n-1}^{r,s} + yV_{n-r-1}^{r,s}, \text{ for } n > r, \tag{2}$$

with initial conditions $V_0^{r,s} = s + 1$ and $V_k^{r,s} = x^k$ for $1 \leq k \leq r$.

For $s = r$ we get the sequence studied by Tuglu et al. [10].

Remark 1.1.

- *For $s = 0$, $(V_n^{r,0})$ is the shifted r-Fibonacci sequence (U_n^r).*
- *For $s = r$, $(V_n^{r,r})$ is the bivariate r–Lucas sequence $L_{r,n}$.*

A few terms of sequences $(V_n^{r,s})$ for $r = 5$ are given in Table 1 below.

$s = 1$	$2, x, x^2, x^3, x^4, x^5, x^6 + 2y, x^7 + 3xy, x^8 + 4x^2y, x^9 + 5x^3y, \ldots$
$s = 2$	$3, x, x^2, x^3, x^4, x^5, x^6 + 3y, x^7 + 4xy, x^8 + 5x^2y, x^9 + 6x^3y, \ldots$
$s = 3$	$4, x, x^2, x^3, x^4, x^5, x^6 + 4y, x^7 + 5xy, x^8 + 6x^2y, x^9 + 7x^3y, \ldots$
$s = 4$	$5, x, x^2, x^3, x^4, x^5, x^6 + 5y, x^7 + 6xy, x^8 + 7x^2y, x^9 + 8x^3y, \ldots$
$s = 5$	$6, x, x^2, x^3, x^4, x^5, x^6 + 6y, x^7 + 7xy, x^8 + 8x^2y, x^9 + 9x^3y, \ldots$

Table 1. Terms of sequences $(V_n^{r,s})$ for $r = 5$

The companion sequences $(V_n^{r,s})$ generalize some classic sequences and polynomials given in Table 2 below; see [10]

x	**y**	**r**	$L_{r,n}(x, y)$
x	y	1	bivariate Lucas polynomials $L_n(x, y)$
x	1	r	Lucas $r-$polynomials $L_{r,n}(x)$
x	1	1	Lucas polynomials $l_n(x)$
1	1	r	Lucas $r-$numbers $L_r(n)$
1	1	1	Lucas numbers L_n
$2x$	y	r	bivariate Pell–Lucas r-polynomials $L_{r,n}(2x, y)$
$2x$	y	1	bivariate Pell–Lucas polynomials $L_n(2x, y)$
$2x$	1	r	Pell–Lucas r-polynomials $Q_{r,n}(x)$
$2x$	1	1	Pell–Lucas polynomials $Q_n(x)$
2	1	1	Pell–Lucas numbers Q_n
$2x$	-1	1	Chebyshev polynomials of the first kind $T_n(x)$
x	$2y$	r	bivariate Jacobsthal–Lucas r-polynomials $L_{r,n}(x, 2y)$
x	$2y$	1	Bivariate Jacobsthal–Lucas polynomials $L_n(x, 2y)$
1	$2y$	1	Jacobsthal–Lucas polynomials $j_n(y)$
1	2	1	Jacobsthal–Lucas numbers j_n

Table 2. Classic sequences and polynomials generalized in terms of $(V_n^{r,s})$

Definition 1.2. *An $n \times n$ matrix $A_n = (a_{ij})$ is called a lower Hessenberg matrix if $a_{ij} = 0$ when $j - i > 1$, i.e.,*

$$A_n = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{bmatrix}. \tag{3}$$

Theorem 1.3. *[2] Let A_n be an $n \times n$ lower Hessenberg matrix for all $n \geq 1$. Then,*

$$\det(A_1) = a_{11}$$

and for $n \geq 2$

$$\det(A_n) = a_{n,n} \det(A_{n-1}) + \sum_{r=1}^{n-1} \left[(-1)^{n-r} a_{n,r} (\prod_{j=r}^{n-1} a_{j,j+1}) \det(A_{r-1}) \right]. \tag{4}$$

Let $A = (a_{ij})$ be an $n \times n$ matrix. The permanent of A, written $\operatorname{per}(A)$, see [8], is defined by

$$\operatorname{per}(A) = \sum_{\sigma} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where the summation extends over all elements σ of the symmetric group S_n.

It is easy to see that,

Remark 1.4. *If A_n is a lower Hessenberg matrix defined in (3), then*

$$\det(A_n) = \operatorname{per}(B_n). \tag{5}$$

where B_n is defined as follows,

$$B_n = \begin{bmatrix} a_{11} & -a_{12} & 0 & \cdots & 0 \\ a_{21} & a_{22} & -a_{23} & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & -a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{bmatrix}.$$

The concept of representing recurrent sequences using special matrices is well known, in particular the representation of the Fibonacci sequence and its generalizations attracted a lot of attention in recent years.

In [7] generalized order k-Fibonacci numbers were represented as permanents of a $(0,1)$-matrix with 1 in position (i, j) for $i - 1 \leq j \leq k + i - 1$ and 0 otherwise. Öcal et al. in 2005 [9] expressed a family of generalized k-Fibonacci numbers as permanent and determinant of special Hessenberg matrices. In [10], determinantal and permanental representations of generalized bivariate r-Fibonacci polynomials were established. In 2012, Kaygisiz and Sahin [6] used the same approach and they gave determinantal and permanental representations of generalized bivariate r-Lucas polynomials.

The aim of this paper is to give the determinantal and permanental representations of the companion sequences family $(V_n^{r,s})$ related to the r-Fibonacci sequence. In the next section, we give two types of order n Hessenberg matrices whose permanent and determinant are the n-th term of the companion sequences family $(V_n^{r,s})$, therefore representations of classic sequences and polynomials can be established by fixing parameters x, y, r and s. In Section 3, we use techniques on the inverse of Hessenberg matrices and our representations introduced in Section 2 to produce n consecutive terms of $(V_n^{r,s})$ simultaneously. Such methods are more convenient compared to the classical recurrent computation in terms of time since one can use properties of Hessenberg matrix inverses along with fast algorithms provided in the literature for this type of matrices; see for example [5], to get n terms of $(V_n^{r,s})$ simultaneously.

2 Main results

We give a determinantal representation of $(V_n^{r,s})$ as follows.

Theorem 2.1. *Let $(V_n^{r,s})$ be the companion sequences family associated to the bivariate r-Fibonacci polynomial and $A_n^{(r,s)} = (a_{jk})$ be an $n \times n$ lower Hessenberg matrix given by*

$$a_{jk} = \begin{cases} i, & \textit{for } j = k-1; \\ x, & \textit{for } j = k; \\ i^r y, & \textit{for } r = j-k \textit{ and } k \neq 1; \\ (s+1)i^r y, & \textit{for } r = j-k \textit{ and } k = 1; \\ 0, & \textit{otherwise;} \end{cases}$$

that is,

$$A_n^{(r,s)} = \begin{bmatrix} x & i & 0 & \cdots & \cdots & & & \cdots & & \cdots & 0 \\ 0 & x & i & 0 & \cdots & & & \ddots & & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \cdots & & & & \cdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \cdots & & & & & 0 \\ (s+1)i^r y & 0 & \cdots & \cdots & & & x & i & 0 & \cdots & 0 \\ 0 & i^r y & 0 & \ddots & & & 0 & x & i & \ddots & 0 \\ \vdots & \ddots & i^r y & 0 & \cdots & & & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \ddots & \ddots & \ddots & & & \ddots & \ddots & x & i \\ 0 & \cdots & & 0 & i^r y & 0 & \cdots & \ddots & \ddots & 0 & x \end{bmatrix}. \tag{6}$$

Then for $n \geq 1$,

$$\det(A_n^{(r,s)}) = V_n^{r,s}. \tag{7}$$

where $i^2 = -1$.

Proof. We will prove Equation (7) by induction over n.

The result is true for $n \in \{1, \dots, r\}$ by hypothesis.

For $n = r+1$, we use Laplace expansion for the last row of $A_{r+1}^{(r,s)}$ then,

$$\det(A_{r+1}^{(r,s)}) = x\det(A_r^{(r,s)}) + (s+1)i^r y \ \det\begin{pmatrix} i & 0 & 0 & \cdots & 0 \\ x & i & 0 & \cdots & 0 \\ \vdots & x & i & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & x & i \end{pmatrix} = x^{r+1} + (s+1)y = V_{r+1}^{r,s}.$$

Assume now that (7) is true for all $j \in \{r+2, \dots, n\}$. Then using Theorem 1.3, we have

$$\begin{aligned}
\det(A_{n+1}^{(r,s)}) &= a_{n+1,n+1}\det(A_n^{(r,s)}) + \sum_{p=1}^{n}\left[(-1)^{n+1-p}a_{n+1,p}(\prod_{j=p}^{n}a_{j,j+1})\det(A_{r-1}^{(r,s)})\right] \\
&= x\det(A_n^{(r,s)}) + \sum_{p=1}^{n-r}\left[(-1)^{n+1-p}a_{n+1,p}(\prod_{j=p}^{n}a_{j,j+1})\det(A_{r-1}^{(r,s)})\right] \\
&\quad + \sum_{p=n-r+1}^{n}\left[(-1)^{n+1-p}a_{n+1,p}(\prod_{j=p}^{n}a_{j,j+1})\det(A_{r-1}^{(r,s)})\right] \\
&= x\det(A_n^{(r,s)}) + \left[(-1)^r(i)^r y\prod_{j=n-r+1}^{n} i\det(A_{n-r}^{(r,s)})\right] \\
&= x\det(A_n^{(r,s)}) + \left[(-1)^r y(i)^r.(i)^r\det(A_{n-r}^{(r,s)})\right] \\
&= x\det(A_n^{(r,s)}) + y\det(A_{n-r}^{(r,s)}).
\end{aligned}$$

From the induction hypothesis and the definition of $V_n^{r,s}$, we obtain

$$\det(A_{n+1}^{(r,s)}) = xV_n^{r,s} + yV_{n-r}^{r,s} = V_{n+1}^{r,s}.$$

Therefore, (7) holds for all positive integers n. $\square$

From Theorem 2.1 and Remark 1.4 we give the permanental representation as follows,

Theorem 2.2. *Let* $(V_n^{r,s})$ *be the the companion sequences family associated to the bivariate* r*-Fibonacci polynomial and* $B_n^{(r,s)} = (b_{jk})$ *be an* $n \times n$ *lower Hessenberg matrix defined by*

$$b_{jk} = \begin{cases} -i, & \text{for } j = k-1; \\ x, & \text{for } j = k; \\ i^r y, & \text{for } r = j-k \text{ and } k \neq 1; \\ (s+1)i^r y, & \text{for } r = j-k \text{ and } k = 1; \\ 0, & \text{otherwise;} \end{cases}$$

that is,

$$B_n^{(r,s)} = \begin{bmatrix} x & -i & 0 & \cdots & \cdots & & & \cdots & & \cdots & 0 \\ 0 & x & -i & 0 & \cdots & & & \ddots & & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \cdots & & & & \cdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \cdots & & & & & 0 \\ (s+1)i^r y & 0 & \cdots & \cdots & & & x & -i & 0 & \cdots & 0 \\ 0 & i^r y & 0 & \ddots & & & 0 & x & -i & \ddots & 0 \\ \vdots & \ddots & i^r y & 0 & \cdots & & & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \ddots & \ddots & \ddots & & & \ddots & \ddots & x & -i \\ 0 & \cdots & & 0 & i^r y & 0 & \cdots & \ddots & \ddots & 0 & x \end{bmatrix}. \tag{8}$$

Then for $n \geq 1$,

$$\operatorname{per}(B_n^{(r,s)}) = V_n^{r,s}. \tag{9}$$

Remark 2.3. *For* $s = r$, $A_n^{(r,r)}$ *is the matrix* $W_{p,n}$ *defined in [6] and* $B_n^{(r,r)}$ *is the matrix* $H_{p,n}$.

Example 2.4. *We give the determinantal and permanental representations of the* 4*-th* $V_n^{r,s}$ *for* $r = 3$ *and* $1 \leq s \leq 3$,

$$\det \begin{bmatrix} x & i & 0 & 0 \\ 0 & x & i & 0 \\ 0 & 0 & x & i \\ 2i^3y & 0 & 0 & x \end{bmatrix} = \operatorname{per} \begin{bmatrix} x & -i & 0 & 0 \\ 0 & x & -i & 0 \\ 0 & 0 & x & -i \\ 2i^3y & 0 & 0 & x \end{bmatrix} = x^4 + 2y = V_4^{3,1}.$$

$$\det \begin{bmatrix} x & i & 0 & 0 \\ 0 & x & i & 0 \\ 0 & 0 & x & i \\ 3i^3y & 0 & 0 & x \end{bmatrix} = \operatorname{per} \begin{bmatrix} x & -i & 0 & 0 \\ 0 & x & -i & 0 \\ 0 & 0 & x & -i \\ 3i^3y & 0 & 0 & x \end{bmatrix} = x^4 + 3y = V_4^{3,2}.$$

$$\det \begin{bmatrix} x & i & 0 & 0 \\ 0 & x & i & 0 \\ 0 & 0 & x & i \\ 4i^3y & 0 & 0 & x \end{bmatrix} = \operatorname{per} \begin{bmatrix} x & -i & 0 & 0 \\ 0 & x & -i & 0 \\ 0 & 0 & x & -i \\ 4i^3y & 0 & 0 & x \end{bmatrix} = x^4 + 4y = V_4^{3,3}.$$

Secondly, we give another type of lower Hessenberg matrices whose determinants and permanents are $V_n^{r,s}$.

Theorem 2.5. *Let* $(V_n^{r,s})$ *be the companion sequences family associated to the bivariate* r*-Fibonacci polynomial and* $C_n^{(r,s)} = (c_{ij})$ *be an* $n \times n$ *lower Hessenberg matrix defined by*

$$c_{ij} = \begin{cases} -1, & \text{for } j = i+1; \\ x, & \text{for } i = j; \\ y, & \text{for } r = i-j \text{ and } j \neq 1; \\ (s+1)y, & \text{for } r = i-j \text{ and } j = 1; \\ 0, & \text{otherwise;} \end{cases}$$

that is,

$$C_n^{(r,s)} = \begin{bmatrix} x & -1 & 0 & \cdots & \cdots & & \cdots & & \cdots & 0 \\ 0 & x & -1 & 0 & \cdots & & \ddots & & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \cdots & & & \cdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \cdots & & & & 0 \\ (s+1)y & 0 & \cdots & \cdots & & & x & -1 & 0 & \cdots & 0 \\ 0 & y & 0 & \ddots & & & 0 & x & -1 & \ddots & 0 \\ \vdots & \ddots & y & 0 & \cdots & & & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \ddots & \ddots & \ddots & & & \ddots & \ddots & x & -1 \\ 0 & \cdots & & 0 & y & 0 & \cdots & \ddots & \ddots & 0 & x \end{bmatrix}. \tag{10}$$

Then,

$$\det(C_n^{(r,s)}) = V_n^{r,s}.$$

Proof. The proof is the same as Theorem 2.1, using recurrence (4). □

From Theorem 2.5 and Remark 1.4 we give the permanental representation as follows.

Theorem 2.6. *Let* $(V_n^{r,s})$ *be the companion sequences family associated to the bivariate* r*-Fibonacci polynomial and* $D_n^{(r,s)} = (d_{ij})$ *be an* $n \times n$ *lower Hessenberg matrix defined by*

$$d_{ij} = \begin{cases} 1, & \textit{for } j = i+1; \\ x, & \textit{for } i = j; \\ y, & \textit{for } r = i-j \textit{ and } j \neq 1; \\ (s+1)y, & \textit{for } r = i-j \textit{ and } j = 1; \\ 0, & \textit{otherwise;} \end{cases}$$

that is,

$$D_n^{(r,s)} = \begin{bmatrix} x & 1 & 0 & \cdots & \cdots & & \cdots & & \cdots & 0 \\ 0 & x & 1 & 0 & \cdots & & \ddots & & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \cdots & & & \cdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \cdots & & & & 0 \\ (s+1)y & 0 & \cdots & \cdots & & & x & 1 & 0 & \cdots & 0 \\ 0 & y & 0 & \ddots & & & 0 & x & 1 & \ddots & 0 \\ \vdots & \ddots & y & 0 & \cdots & & & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \ddots & \ddots & \ddots & & & \ddots & \ddots & x & 1 \\ 0 & \cdots & & 0 & y & 0 & \cdots & \ddots & \ddots & 0 & x \end{bmatrix}. \tag{11}$$

Then,

$$\operatorname{per}(D_n^{(r,s)}) = V_n^{r,s}.$$

Example 2.7. *We give the determinantal and permanental representations of the* 4*-th* $V_n^{r,s}$ *for* $r = 3$ *and* $1 \leq s \leq 3$ *using matrices* $C_n^{(r,s)}$ *and* $D_n^{(r,s)}$*,*

$$\det \begin{bmatrix} x & -1 & 0 & 0 \\ 0 & x & -1 & 0 \\ 0 & 0 & x & -1 \\ 2y & 0 & 0 & x \end{bmatrix} = \text{per} \begin{bmatrix} x & 1 & 0 & 0 \\ 0 & x & 1 & 0 \\ 0 & 0 & x & 1 \\ 2y & 0 & 0 & x \end{bmatrix} = x^4 + 2y = V_4^{3,1}.$$

$$\det \begin{bmatrix} x & -1 & 0 & 0 \\ 0 & x & -1 & 0 \\ 0 & 0 & x & -1 \\ 3y & 0 & 0 & x \end{bmatrix} = \text{per} \begin{bmatrix} x & 1 & 0 & 0 \\ 0 & x & 1 & 0 \\ 0 & 0 & x & 1 \\ 3y & 0 & 0 & x \end{bmatrix} = x^4 + 3y = V_4^{3,2}.$$

$$\det \begin{bmatrix} x & -1 & 0 & 0 \\ 0 & x & -1 & 0 \\ 0 & 0 & x & -1 \\ 4y & 0 & 0 & x \end{bmatrix} = \text{per} \begin{bmatrix} x & 1 & 0 & 0 \\ 0 & x & 1 & 0 \\ 0 & 0 & x & 1 \\ 4y & 0 & 0 & x \end{bmatrix} = x^4 + 4y = V_4^{3,3}.$$

Remark 2.8. *Our results are valid for all classic sequences and polynomials stated in Table 2 by fixing parameters* x*,* y*,* r *and* s*.*

3 Applications

In this section determinantal and permanental representations of $V_n^{r,s}$ are used to give n consecutive terms of $V_n^{r,s}$ simultaneously.

In [3], the authors gave a new method to compute the inverse and the determinant of a Hessenberg matrix. They defined a triangular matrix $\widetilde{H}$ associated with a Hessenberg matrix H as follows,

$$\widetilde{H} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ h_{11} & h_{12} & 0 & \ddots & \vdots & 0 \\ h_{21} & h_{22} & h_{23} & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ h_{n-1,1} & h_{n-1,2} & \cdots & h_{n-1,n-1} & h_{n-1,n} & 0 \\ h_{n,1} & h_{n,2} & \cdots & h_{n,n-1} & h_{n,n} & 1 \end{bmatrix}.$$

Then partitioning $\widetilde{H}^{-1}$ into

$$\widetilde{H}^{-1} = \begin{bmatrix} \alpha & \mathbf{L} \\ h & \beta^T \end{bmatrix},$$

where $\alpha, \mathbf{L}, h$ and β^T are matrices of size $n \times 1, n \times n, 1 \times 1, n \times 1$, respectively. They obtained the following equalities,

$$det(H) = (-1)^n h.det(\widetilde{H}), \tag{12}$$

and,

$$H\alpha + he_n = 0. \tag{13}$$

Now we present our results.

Theorem 3.1. *Let $\widetilde{A}_{n+1}^{(r,s)}$ be the $(n+1)\times(n+1)$ non-singular matrix defined by*

$$\widetilde{A}_{n+1}^{(r,s)} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ & & & & & \vdots \\ & & A_n^{(r,s)} & & & 0 \\ & & & & & 0 \\ & & & & & 1 \end{bmatrix},$$

where $A_n^{(r,s)}$ is the n-th order matrix of defined in Theorem 2.1. Then the first column of $(\widetilde{A}_{n+1}^{(r,s)})^{-1}$ is as follows,

$$\begin{bmatrix} 1 \\ iV_1^{r,s} \\ i^2V_2^{r,s} \\ \vdots \\ i^{n-1}V_{n-1}^{r,s} \\ i^{n+1}V_n^{r,s} \end{bmatrix},$$

where $i^2 = -1$.

Proof. Following the decomposition approach used in [3], we construct $(\widetilde{A}_{n+1}^{(r,s)})^{-1}$ as follows,

$$(\widetilde{A}_{n+1}^{(r,s)})^{-1} = \left[\begin{array}{c|c} (\alpha)_{n\times 1} & (L)_{n\times n} \\ \hline (h)_{1\times 1} & (\beta^T)_{1\times n} \end{array}\right],$$

Hence the first column of $(\widetilde{A}_{n+1}^{(r,s)})^{-1}$ is

$$\left[\begin{array}{c} (\alpha)_{n\times 1} \\ \hline h \end{array}\right]_{(n+1)\times 1}.$$

Then from Theorem 2.1 and Equalities (12) and (13) we have

$$\det(A_n^{(r,s)}) = (-1)^n h.\det(\widetilde{A}_{n+1}^{(r,s)}) \Rightarrow h = \frac{\det(A_n^{(r,s)})}{(-1)^n \det(\widetilde{A}_{n+1}^{(r,s)})} = \frac{V_{n+1}^{r,s}}{(-1)^n (i)^{n-1}} = i^{n+1}V_n^{r,s}$$

and

$$[\alpha] = -(A_n^{(r,s)})^{-1}(i)^{n+1}V_n^{r,s}e_n = \begin{bmatrix} 1 \\ iV_1^{r,s} \\ i^2V_2^{r,s} \\ \vdots \\ i^{n-1}V_{n-1}^{r,s} \end{bmatrix}.$$

Consequently, we get the desired result. □

Theorem 3.2. *Let $\widetilde{C}_{n+1}^{(r,s)}$ be the $(n+1)\times(n+1)$ non-singular matrix defined by*

$$\widetilde{C}_{n+1}^{(r,s)} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ & & & & & \vdots \\ & & C_n^{(r,s)} & & & 0 \\ & & & & & 0 \\ & & & & & 1 \end{bmatrix},$$

where $C_n^{(r,s)}$ is the n-th order matrix of defined in Theorem 2.5. Then the first column of $(\widetilde{C}_{n+1}^{(r,s)})^{-1}$ is as follows,

$$\begin{bmatrix} 1 \\ V_1^{r,s} \\ V_2^{r,s} \\ \vdots \\ V_{n-1}^{r,s} \\ -V_n^{r,s} \end{bmatrix}.$$

Proof. The proof is the same as Theorem 3.1. □

4 Conclusion

We give two types of determinantal and permanental representations of companion sequences associated to the r-Fibonacci sequence, our results generalize the representations of generalized bivariate r-Lucas polynomials given in [6].

Hence, determinantal and permanental representations of companion sequences associated to classic Lucas numbers L_n, r-Lucas polynomials $L_{r,n}(x)$, $\ldots$ are established.

As applications of our determinantal and permanental representations, we get n consecutive terms of companion sequences simultaneously in the first column of the inverse of a special Hessenberg matrix.

Similar representations to those provided in Section 2 could be established by alternative methods using generating functions of $(V_n^{r,s})$, for instance Wronski's formula; see [4, page 17], is used to produce the coeffcients of the reciprocal of a formal series in terms of determinants of Toeplitz–Hessenberg matrices. Also the result obtained in Section 3 using Hessenberg matrices can be obtained using inverses of triangular Toeplitz matrices with modified first column.

Acknowledgment

The authors gratefully acknowledge the referees for valuable comments and suggestions. This work was supported partially by DGRSDT Grant number C0656701.

References

[1] Abbad, S., Belbachir, H., & Benzaghou, B. (2019). Companion sequences associated to the r-Fibonacci sequence: Algebraic and combinatorial properties. *Turkish Journal of Mathematics*, 43(3), 1095–1114.

[2] Cahill, N. D., D'Errico, J. R., Narayan, D. A., & Narayan, J. Y. (2002). Fibonacci determinants. *The College Mathematics Journal*, 33(3), 221–225.

[3] Chen, Y.-H., & Yu, C.-Y. (2011). A new algorithm for computing the inverse and the determinant of a Hessenberg matrix. *Applied Mathematics and Computation*, 218(8), 4433–4436.

[4] Henrici, P. (1974). *Applied and Computational Complex Analysis* (Vol. 1). Wiley.

[5] Ikebe, Y. (1979). On inverses of Hessenberg matrices. *Linear Algebra and Its Applications*, 24, 93–97.

[6] Kaygisiz, K., & Sahin, A. (2012). Generalized bivariate Lucas p-polynomials and Hessenberg Matrices. *Journal of Integer Sequences*, 15, Article 12.3.4.

[7] Minc, H. (1964). Permanents of $(0, 1)$-circulants. *Canadian Mathematical Bulletin*, 7(2), 253–263.

[8] Minc, H. (1984). *Permanents* (Vol. 6). Cambridge University Press.

[9] Öcal, A. A., Tuglu, N., & Altinişik, E. (2005). On the representation of k-generalized Fibonacci and Lucas numbers. *Applied Mathematics and Computation*, 170(1), 584–596.

[10] Tuglu, N., Kocer, E. G., & Stakhov, A. (2011). Bivariate Fibonacci like p-polynomials. *Applied Mathematics and Computation*, 217(24), 10239–10246.