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Factorials as repdigits in base b

Nurettin Irmak[1] and Alain Togbé[2]

[1] Department of Fundamental Science, Faculty of Engineering and Natural Science,
Konya Technical University, Konya, Turkey
e-mails: `irmaknurettin@gmail.com`, `nirmak@ktun.edu.tr`

[2] Department of Mathematics, Purdue University Northwest
1401 S. U. S. 421., Westville, IN 46391, United States
e-mail: `atogbe@pnw.edu`

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Abstract: Let $b \in \{2, 3, \ldots, 9\}$. In this paper, we show that the solutions of the equation $(x)_b = m!$ are $(11)_5 = 3!$, $(33)_7 = (44)_5 = 4!$, where $(x)_b$ has at least two digits.
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1 Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_n = F_{n-1} + F_{n-2}$, for $n \geq 0$, with $F_0 = 0$ and $F_1 = 1$. The first few terms of this sequence are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots.$$

The following Table 1 shows several generalizations of the Fibonacci sequence.

Finding special properties in these sequences is a very interesting problem. A number of mathematicians studied equations involving the above sequences, repdigits and factorials. For example,

- Marques and Lengyel [8] solved the equation $T_n = m!$.
- Irmak [3] found the solutions of the equation $w_n = m!$, where w_n is the n-th term of Perrin or Padovan sequence.

Sequence	Recurrence relation	Initial conditions
Lucas	$L_n = L_{n-1} + L_{n-2}$	$L_0 = 2$ and $L_1 = 1$
Pell	$P_n = 2P_{n-1} + P_{n-2}$	$P_0 = 0$ and $P_1 = 1$
Pell–Lucas	$Q_n = 2Q_{n-1} + Q_{n-2}$	$Q_0 = 2$ and $Q_1 = 2$
Balancing	$B_n = 6B_{n-1} - B_{n-2}$	$B_0 = 0$ and $B_1 = 1$
Jacobshtal	$J_n = J_{n-1} + 2J_{n-2}$	$J_0 = 0$ and $J_1 = 1$
Tribonacci	$T_n = T_{n-1} + T_{n-2} + T_{n-3}$	$T_0 = 0,\ T_1 = 1$, and $T_2 = 1$
Perrin	$R_n = R_{n-2} + R_{n-3}$	$R_0 = 3$, $R_1 = 0$, and $R_2 = 2$
Padovan	$P_n = P_{n-2} + P_{n-3}$	$P_0 = 1$, $P_1 = 1$, and $P_2 = 1$.

Table 1. Generalizations of the Fibonacci sequence

- Luca [5] found repdigits in the Fibonacci and Lucas sequences.
- Marques and Togbé [9] handled the equation

$$F_nF_{n+1}\dots F_{n+k-1} = d\left(\frac{10^m - 1}{9}\right).$$

- The equation

$$L_nL_{n+1}\dots L_{n+k-1} = d\left(\frac{10^m - 1}{9}\right)$$

  was solved by Irmak and Togbé [4].

There are also several results including sum of the members of linear recurrences (given in the table) which are repdigits (see the papers [1, 2, 10–14]).

Motivated by these papers, it is natural to ask the following question:

What factorials are repdigits in base b?

We answer this question by proving the following theorem.

Theorem 1.1. *Let $b \in \{2, 3, \dots, 9\}$ and x, m be positive integers. The solutions of the equation*

$$(x)_b = m!$$

are given by $(11)_5 = 3!,\ (33)_7 = (44)_5 = 4!$.

To prove this theorem, we will characterize and use the 2-adic values $\nu_2\left(b^n - 1\right)$. The p-adic order, $\nu_p(r)$ of r is the exponent of the highest power of a prime p which divides r.

2 Auxiliary results

Now, we will give the 2-adic order of the term $b^n - 1$, for $b \in \{2, 3, \dots, 9\}$ by proving the following theorem. It is obvious that $\nu_2(b^k - 1) = 0$ for even b.

Theorem 2.1. *For $k \geq 1$, we have*

(i)

$$\nu_2\left(3^k-1\right)=\begin{cases}\nu_2(k)+2, & \text{if } k \text{ is even}\\ 1, & \text{if } k \text{ is odd}\end{cases}$$

(ii)

$$\nu_2\left(5^k-1\right)=\begin{cases}\nu_2(k)+2, & \text{if } k \text{ is even}\\ 1, & \text{if } k \text{ is odd}\end{cases}$$

(iii)

$$\nu_2\left(7^k-1\right)=\begin{cases}\nu_2(k)+1, & \text{if } k \text{ is even}\\ 1, & \text{if } k \text{ is odd}\end{cases}$$

(iv)

$$\nu_2\left(9^k-1\right)=\begin{cases}\nu_2(k)+3, & \text{if } k \text{ is even}\\ 1, & \text{if } k \text{ is odd.}\end{cases}$$

Proof. Firstly, we will deal with the first 2-adic order. Assume that k is even positive integer. To prove it, we need to show that

$$3^{2^k s}-1 \equiv 2^{k+2}s \pmod{2^{k+3}}.$$

We will use the induction on s to prove the congruence. Firstly, we will deal with the basic case $s=1$. So, we want to prove that

$$3^{2^k}-1 \equiv 2^{k+2} \pmod{2^{k+3}}.$$

Now, we will use the induction on k. Obviously, the congruence holds for $k=1$. Then we suppose that the congruence

$$3^{2^k}-1 \equiv 2^{k+2} \pmod{2^{k+3}} \tag{1}$$

is true for k. Our aim is to show that $3^{2^{k+1}}-1 \equiv 2^{k+3} \pmod{2^{k+4}}$. The congruence (1) implies that

$$3^{2^k}=2^{k+2}+1+l_1 2^{k+3}, \tag{2}$$

for some l_1. So, we deduce that

$$\begin{aligned} 3^{2^{k+1}} &= \left(3^{2^k}\right)^2=\left(2^{k+2}+1+l_1 2^{k+3}\right)^2 \\ &= 2^{2(k+3)}l_1^2+2^{2(k+3)}l_1+2^{2(k+2)}+2^{k+4}l_1+2^{k+3}+1 \end{aligned}$$

and this gives

$$3^{2^{k+1}}-1 \equiv 2^{k+3} \pmod{2^{k+4}}$$

as desired. Here, we used the facts that $2k+6 \geq k+3$ and $2k+4 \geq k+3$, for $k \geq 1$.

By the induction hypothesis, $3^{2^k s}-1 \equiv 2^{k+2}s \pmod{2^{k+3}}$ holds for s. It means that there exists the integer l_2 such that $3^{2^k s}=2^{k+2}s+l_2 2^{k+2}+1$. From this and (2), we deduce that

$$\begin{aligned} 3^{2^k(s+1)} &= 3^{2^k s}\cdot 3^{2^k} \\ &= \left(2^{k+2}s+l_2 2^{k+2}+1\right)\left(2^{k+2}+1+l_1 2^{k+3}\right). \end{aligned}$$

So,

$$3^{2^k(s+1)} - 1 \equiv 2^{k+2}(s+1) \pmod{2^{k+3}}$$

follows.

From now on, suppose that k is odd. Our aim is to show that

$$3^{2w+1} - 1 \equiv 2 \pmod 4. \tag{3}$$

We use the induction method again. It is easy to see that the congruence holds for $w = 1$. Then assume that it is true for w. If we multiply the congruence (3) with 9, then the congruence

$$3^{2w+3} - 1 \equiv 2 \pmod 4$$

holds as claimed. This finished the proof. The remaining items can be similarly proven. Therefore, we leave the details to the reader. □

The following lemma gives the upper and lower bounds for the term $\nu_p(k!)$. To prove this, we refer to Lemma 2.2 in [7].

Lemma 2.1. *For any integer $k \geq 1$ and p prime, we have*

$$\frac{k}{p-1} - \left\lfloor \frac{\log k}{\log p} \right\rfloor \leq \nu_p(k!) \leq \frac{k-1}{p-1},$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

3 Proof of Theorem

Assume that $(x)_b = d\left(\frac{b^k-1}{b-1}\right)$ for $d \in \{1, 2, \ldots, 9\}$ and $b \in \{2, \ldots, 9\}$. By using Lemma 2.1 with Theorem 2.1, we have

$$\begin{aligned} m - \left\lfloor \frac{\log m}{\log 2} \right\rfloor &\leq \nu_2(m!) = \nu_2\left(d\frac{b^k-1}{b-1}\right) \\ &\leq \nu_2(k) + 3 + \nu_2(d) \leq \nu_2(k) + 6. \end{aligned}$$

It means that $2^{m-6-\left\lfloor \frac{\log m}{\log 2} \right\rfloor}$ divides k. Then

$$2^{m-6-\left\lfloor \frac{\log m}{\log 2} \right\rfloor} \leq k \tag{4}$$

follows. It is known that $m! < \left(\frac{m}{2}\right)^m$. This fact gives

$$(k-1)\log b < \log\left(d\frac{b^k-1}{b-1}\right) < m\left(\log\frac{m}{2}\right). \tag{5}$$

Combining (4) and (5), we arrive at

$$2^{m-6-\left\lfloor \frac{\log m}{\log 2} \right\rfloor} < \frac{m\log\left(\frac{m}{2}\right)}{\log b} + 1.$$

This inequality implies that $m \leq 14$ for $b \in \{2, \ldots, 9\}$. We use a simple routine written in *Mathematica* which gives the solutions as listed in Theorem 1.1. The proof of our main result is complete. □

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