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On certain inequalities for the prime counting function

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Abstract: We study certain inequalities for the prime counting function $\pi(x)$. Particularly, a new proof and a refinement of an inequality from [1] is provided.
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1 Introduction

Let $\pi(x)$ denote the number of primes $\leq x$, where $x \geq 1$ is an integer. The famous Hardy–Littlewood conjecture states that the inequality

$$\pi(x+y) \leq \pi(x) + \pi(y) \tag{1}$$

is valid for all $x, y \geq 2$. Neither a proof nor a counterexample is known up to now.

There exist many inequalities in the literature, related to (1). For a survey of results, see the recent paper [1] of the author and H. Alzer and M. K. Kwong.

Many earlier results on $\pi(x)$ can be found in Chapter VII of the monograph [4]. For connections with other arithmetic functions, see the recent book [5] (see pp. 159–160).

One of the main results, proved in [1], is the inequality (see Theorem 1 of [1])

$$\pi^2(x+y) \geq \frac{16}{9}\pi(x).\pi(y), \tag{2}$$

where $x, y \geq 2$ and with equality only for $x = y = 5$.

For $x = y$ in (2), we get the result that

$$\pi(2x) \geq \frac{4}{3}.\pi(x), \tag{3}$$

with equality only for $x = 5$. This is a converse of the famous Landau inequality

$$\pi(2x) \leq 2\pi(x), \qquad (x \geq 2). \tag{4}$$

Another result of [1] is the following (see left-hand side of Theorem 6 of [1]):

$$\frac{1}{2} \leq \frac{\pi(x)^{x/(x+y)}.\pi(y)^{y/(x+y)}}{\pi(x+y)}. \tag{5}$$

The aim of this paper is to prove that, a converse inequality of (1) holds true, and this gives a new proof, as well as a refinement of (2). Another result will be motivated by relation (5).

2 Main results

The following classical inequality due to Rosser and Schoenfeld [3] will be used:
Lemma For all $x \geq 67$ one has

$$\frac{x}{\log x - \frac{1}{2}} < \pi(x) < \frac{x}{\log x - \frac{3}{2}}. \tag{6}$$

The first main result of this paper gives a converse to inequality (1):

Theorem 1. *For all $x, y \geq 2$ one has*

$$\pi(x+y) \geq \tfrac{2}{3}.[\pi(x) + \pi(y)], \tag{7}$$

with equality only for $(x, y) = (5, 5); (3, 7); (7, 3)$.

Proof. Let $f(x) = \frac{x}{\log x - \frac{3}{2}}$. We shall prove that, this function is strictly concave for $x > e^{\frac{7}{2}}$. Indeed, one has $f'(x) = (\log x - \frac{5}{2})/(\log x - \frac{3}{2})^2$, and after some elementary computations, we get $f''(x).x.(\log x - \frac{3}{2})^3 = -\log x + \frac{7}{2} < 0$ if $\log x > \frac{7}{2}$, i.e., $x > e^{\frac{7}{2}} \approx 33.11\ldots$.

The concavity of $f(x)$ gives the inequality:

$$f(x) + f(y) \leq 2f(\tfrac{x+y}{2}) \text{ for all } x, y \geq e^{\frac{7}{2}}. \tag{8}$$

By the right-hand side of (6) and (8) we can write:

$$\pi(x) + \pi(y) < f(x) + f(y) \leq \frac{x+y}{\log(\frac{x+y}{2}) - \frac{3}{2}}. \tag{9}$$

Now, by the left-hand side of (6) one has $\frac{3}{2}\pi(x+y) > \frac{3}{2}.\frac{x+y}{\log(x+y)-\frac{1}{2}}$, so at a first step, in attempt to have (7), we want to prove the inequality:

$$\frac{x+y}{\log(\frac{x+y}{2}) - \frac{3}{5}} < \frac{3}{2}.\frac{x+y}{\log(x+y) - \frac{1}{2}}, \tag{10}$$

which is equivalent with

$$\log(x+y) > 3\log 2 + \frac{9}{5} - \frac{1}{2} = 3.379\ldots,$$

i.e., $x+y > e^{3.379\ldots} \approx 29.3\ldots$.

This is clearly true, if $x, y \geq 67$. Therefore, inequality (7) is proved for all $x, y \geq 67$.

Now, suppose that $x \geq y$ and $y \leq 66$. Then $\pi(y) \leq 18$, so $\frac{2}{3}.[\pi(x)+\pi(y)] \leq \frac{2}{3}.[\pi(x)+18] = \frac{2}{3}\pi(x)+12$. We have to prove that $\frac{2}{3}\pi(x)+12 \leq \pi(x+y)$, or

$$2\pi(x) + 36 \leq 3\pi(x+y). \tag{11}$$

As $3\pi(x) \leq 3\pi(x+y)$, it will be sufficient to consider the inequality $2\pi(x)+36 \leq 3\pi(x)$, i.e., $\pi(x) \geq 36$. This is true, if $x \geq 151$.

Finally, we have to verify the case:

$$2 \leq y \leq x \leq 150, y \leq 66. \tag{12}$$

This can be verified by a computer (for example, a Maple 13 program). This finishes the proof of Theorem 1. □

Corollary 1.

$$\pi^2(x+y) \geq \frac{4}{9}.[\pi(x)+\pi(y)]^2 \geq \frac{16}{9}\pi(x).\pi(y), \tag{13}$$

which is a refinement of inequality (2).

Remark 1. *For $y \leq x$ there is equality in the first inequality of (13) for $y = 3, x = 7$ and $y = 5, x = 5$; while in the second inequality only for $y = 5, x = 5$.*

Indeed, the first inequality follows by (7), while the second one by $(a+b)^2 \geq 4a.b$, where $a = \pi(x), b = \pi(y)$.

Now, by the weighted arithmetic mean—geometric mean inequality one has:

$$u^\alpha.v^\beta \leq \alpha.u + \beta.v \tag{14}$$

for $u, v, \alpha, \beta > 0; \alpha+\beta = 1$. By letting $u = \pi(x)$, $\alpha = x/(x+y)$, $v = \pi(y)$, $\beta = y/(x+y)$, by (5) and (14) we get

$$\pi(x+y) \leq 2.\pi(x)^{x/(x+y)}.\pi(y)^{y/(x+y)} \leq 2[\frac{x}{x+y}\pi(x) + \frac{y}{x+y}\pi(y)],$$

i.e.,

$$(x+y).\pi(x+y) \leq 2.[x.\pi(x) + y.\pi(y)]. \tag{15}$$

In 2001, Panaitopol [2] proved the inequality:

$$\pi^2(x+y) \leq 2.[\pi^2(x) + \pi^2(y)]. \tag{16}$$

Motivated by these two inequalities, in what follows, we shall prove:

Theorem 2. *For all $x, y \geq 2$ one has*

$$\pi^2(x+y) \leq \frac{8}{7}.[x.\pi(x) + y.\pi(y)], \tag{17}$$

with equality only for $(x, y) = (3, 4); (4, 3)$.

Proof. Let us consider the function $g(x) = \dfrac{x^2}{\log x - \frac{1}{2}}$ $(x > 0)$. After elementary computations we can deduce that

$$\frac{1}{2}.g''(x).\left(\log x - \frac{1}{2}\right)^2 = \log^2 x - \frac{3}{2}\log x + 1. \tag{18}$$

Letting $\log x = t$, clearly $t^2 - \frac{3}{2}t + 1 > 0$ (having a negative discriminant), so we get that the function $g(x)$ is strictly convex.

By the left-hand side of (6) one has

$$x\pi(x) + y\pi(y) > g(x) + g(y) \geq 2g\left(\frac{x+y}{2}\right) = \left(\frac{x+y}{2}\right)^2 \Big/ \left(\log\left(\frac{x+y}{2}\right) - \frac{1}{2}\right),$$

by the convexity of $g(x)$.

By the right-hand side of (6), in order to prove (17), we have first to consider the validity of inequality

$$\frac{8}{7}.\frac{(x+y)^2}{4.[\log(x+y) - \log 2 - \frac{1}{2}]} > \frac{(x+y)^2}{(\log(x+y) - \frac{3}{2})^2}. \tag{19}$$

Letting $\log(x+y) = m$, this becomes after elementary computations:

$$2m^2 - 13m + 7\log^2 + 8 > 0.$$

Solving this quadraatic inequality, it follows that it is true for $m > 2.64\ldots$, i.e., $x + y > e^{2.64\ldots} = 14.01\ldots$, which is clearly true for $x, y \geq 67$.

Now, let $x \geq y$ and $y \leq 66$. As $y\pi(y) \geq 2$, it is sufficient to consider the inequality:

$$(\pi(x) + 18)^2 \leq \frac{8}{7}.[x\pi(x) + 2]. \tag{20}$$

This can written as $7\pi^2(x) + 252\pi(x) + 2268 \leq 8x\pi(x) + 16$. Now $8x\pi(x) \geq 12\pi^2(x)$ by the elementary inequality

$$\frac{\pi(x)}{x} \leq \frac{2}{3}x, \qquad (x \geq 2). \tag{21}$$

Therefore, we have to consider

$$5\pi^2(x) - 252\pi(x) - 2252 \geq 0,$$

which is valid for $\pi(x) \geq 38$, i.e., $x \geq 163$.

It remains to verify inequality (17) for

$$2 \leq y \leq x \leq 163. \tag{22}$$

This can be verified by a computer, but we can could reduce the numbers of verifications as follows:

Segal [6] proved in 1962 that inequality (1) holds true for any $x, y \geq 2$ and $x + y \leq 101081$.

Thus we can write for the values from (22) that

$$7\pi^2(x + y) \leq 7\pi^2(x) + 7\pi^2(y) + 14\pi(x).\pi(y). \tag{23}$$

Now, if we can prove that $8x\pi(x) \geq 14\pi^2(x)$, then we would have $8x\pi(x) + 8y\pi(y) \geq 14\pi^2(x) + 14\pi(y)$ and inequality (23) would follow on base of $7a^2 + 7b^2 > 14ab$ (i.e., $7(a-b)^2 > 0$) for $a = \pi(x), b = \pi(y)$.

The inequality $8x\pi(x) \geq 14\pi^2(x)$ is in fact

$$\pi(x) \leq \frac{4}{7}x, \tag{24}$$

which is similar to (21), and is valid for all $x \geq 6$.

This is a simple exercise, so (22) can be reduced to

$$2 \leq y \leq x \leq 5. \tag{25}$$

For these cases, even a verification by hand can be done. This finishes the proof of (17). □

Remark 2. *The constants* $2/3$ *and* $8/7$ *in Theorems 1 and 2 are the best possible. In a forthcoming paper some other inequalities of a new type will be presented.*

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