Formulas for the n-th prime number

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Abstract: A short review of formulas for the n-th prime number is given and some new formulas are introduced.
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1 Introduction

The work on the present paper started around 2015. Section 3 was completed in the middle of 2018, when I decided that it will be better if there is a computational evaluation of the effectiveness of the formulas for the n-th prime number p_n, described in this paper. I asked my former PhD student—now Associate Professor—Dimitar Dimitrov to implement in software this idea and make a detailed comparison of the computation times that different existing formulae require to compute the n-th prime. His work resulted in his publication [10], where my research was referenced as "unpublished manuscript". Dimitrov found in the literature and included in his research, four new formulas—three of S. M. Ruiz and one of I. Kaddoura and S. Abdul-Nabi—which I also am adding in my literature review in Section 2.

While the motivation of his paper was that "unpublished manuscript" of mine, back in that moment in 2019 I was working on other research problems and I delayed the present publication hoping that some other formulas are on the way. By these reasons, only now, two years later, I have revisited my research and I have added four new formulas in Section 4.

2 Some formulas for the n-th prime number

Although the problem for finding an explicit formula for the n-th prime number is very old, it obtained solutions even in the second part of the last century.

Probably, the first explicit formula giving the n-th prime number p_n was introduced in 1962 by L. Veshenevskiy in [21]. It has the form:

$$p_n = 2 + \frac{1 + (-1)^{2^{n-k_2}}}{2} + \frac{1 + (-1)^{2^{(n-k_2).(n-k_2-k_3)}}}{2} + \frac{1 + (-1)^{2^{(n-k_2).(n-k_2-k_3).(n-k_2-k_3-k_4)}}}{2} + \ldots$$

where

$$k_2 = \frac{1 + (-1)^{2^{(2-2.2)^2}}}{2},$$

$$k_3 = \frac{1 + (-1)^{2^{(3-2.2)^2.(3-2.3)^2.(3-3.2)^2}}}{2},$$

$$\ldots$$

$$k_t = \frac{1 + (-1)^{2^{(t-2.2)^2.(t-2.3)^2\ldots(t-2.t)^2(t-3.2)^2.(t-3.3)^2\ldots(t-3.(t-1))^2\ldots(t-m.2)^2.(t-m.3)^2\ldots(t-m.(t-m+2))^2\ldots(t-t.2)^2}}}{2}.$$

In Paulo Ribenboim's book [16], three other formulas for p_n are discussed. The first of them is introduced in 1964 by C. P. Willans [22]. For every integer $j \geq 1$ let

$$F(j) = \left[\cos^2 \pi \frac{(j-1)! + 1}{j} \right]$$

and

$$H(j) = \left[\frac{\sin^2 \frac{\pi\{(j-1)!\}^2}{j}}{\sin^2\left(\frac{\pi}{j}\right)} \right],$$

where $[x]$ is the integer part of the real number x.

Let π be the well-known prime counting function. Then it satisfies the equalities

$$\pi(n) = -1 + \sum_{j=1}^{n} F(j)$$

and

$$\pi(n) = \sum_{j=2}^{n} H(j).$$

Hence, for the natural number $n \geq 2$

$$p_n = 1 + \sum_{m=1}^{2^n} \left[\left[\frac{n}{\sum_{j=1}^{m} F(j)} \right]^{1/n} \right]$$

$$= 1 + \sum_{m=1}^{2^n} \left[\left[\frac{n}{1 + \pi(m)} \right]^{1/n} \right] = 1 + \sum_{m=1}^{2^n} \left[\left[\frac{n}{1 + \sum_{j=2}^{m} H(j)} \right]^{1/n} \right].$$

In 1971 J. M. Gandhi [11] introduced the recurrent formula

$$p_n = \left[1 - \frac{1}{\log 2} \log \left(-\frac{1}{2} + \sum_{d|P_{n-1}} \frac{\mu(d)}{2^d - 1} \right) \right],$$

where μ is the well-known Möbius function and $P_{n-1} = p_1 \ldots p_{n-1}$, with $p_1, \ldots, p_{n-1}$ already calculated prime numbers.

Following [16], we note that J. Mináč in an unpublished paper gave the formula

$$\pi(n) = \sum_{j=2}^{n} \left[\frac{(j-1)! + 1}{j} - \left[\frac{(j-1)!}{j} \right] \right].$$

Now, for p_n, e.g., Willans formula can be used with the new form of $\pi(n)$.

In 2000 and 2005, S. M. Ruiz introduced the following three formulas [17, 18]:

$$p_n = 1 + \sum_{k=1}^{[2n \log n + 2]} \left(1 - \left[\frac{1}{n} \sum_{j=2}^{k} \left(1 + \left[\frac{1}{j} \left(2 - \sum_{s=1}^{j} \left(\left[\frac{j}{s} \right] - \left[\frac{j-1}{s} \right] \right) \right) \right] \right) \right] \right)$$

$$p_n = 1 + \sum_{k=1}^{[2n \log n + 2]} \left(1 - \left[\frac{1}{n} \sum_{j=2}^{k} \left[\frac{\mathrm{LCM}(1, 2, \ldots, j)}{j. \, \mathrm{LCM}(1, 2, \ldots, j-1)} \right] \right] \right),$$

$$p_n = [n \log n] + \sum_{k=[n \log n]}^{[n \log n + n(\log(\log n) - 0.5) + 3]} \left(1 - \left[\frac{1}{n} \sum_{j=2}^{k} \left[\frac{\mathrm{LCM}(1, 2, \ldots, j)}{j. \, \mathrm{LCM}(1, 2, \ldots, j-1)} \right] \right] \right),$$

and in 2012, I. Kaddoura and S. Abdul-Nabi in [12] give the formula:

$$p_n = 3 + 2[n \log n] - \sum_{x=7}^{[2n \log n + 2]} \left[\frac{4 + \sum_{j=1}^{} \left[\frac{x-1}{6} \right] [S(6j+1)] + \sum_{j=1}^{} \left[\frac{x+1}{6} \right] [S(6j-1)]}{n} \right],$$

where

$$S(x) = - \frac{\sum_{k=1}^{\left[\frac{[\sqrt{x}]}{6} \right] + 1} \left(\left[\left[\frac{x}{6k+1} \right] - \frac{x}{6k+1} \right] + \left[\left[\frac{x}{6k-1} \right] - \frac{x}{6k-1} \right] \right)}{2 \left(\left[\frac{[\sqrt{x}]}{6} \right] + 1 \right)}.$$

In [3, 8, 9], the author introduced four other explicit formulas, giving the n-th prime number. First, let us define functions sg, $\overline{\mathrm{sg}}$ and fr by:

$$\mathrm{sg}(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ 1, & \text{if } x > 0 \end{cases}, \quad \overline{\mathrm{sg}}(x) = \begin{cases} 0, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases},$$

where x is a real number and

$$\mathrm{fr}\left(\frac{p}{q} \right) = \begin{cases} 0, & \text{if } p = 1 \\ 1, & \text{if } p \neq 1 \end{cases},$$

where p and q are natural numbers, such that $(p, q) = 1$.

Let the natural number $n > 1$ have a canonical representation

$$n = \prod_{i=1}^{k} p_i^{\alpha_i}, \tag{1}$$

where $p_1, \ldots, p_k$ are distinct prime numbers and $\alpha_1, \ldots, \alpha_k \geq 1$ are natural numbers.

The well-known arithmetic functions φ, ψ and σ (see, e.g., [14, 15]) have the forms:

$$\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1), \ \varphi(1) = 1,$$

$$\psi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i + 1), \ \psi(1) = 1,$$

$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i + 1} - 1}{p_i - 1}, \ \sigma(1) = 1.$$

The following four representations of function π hold for every natural number $n \geq 2$ with canonical form (1) (here and below, the index of the function π corresponds to the function, that is used in the representation):

$$\pi_\varphi(n) = \sum_{j=2}^{n} \overline{sg}(j - 1 - \varphi(j));$$

$$\pi_\psi(n) = \sum_{j=2}^{n} \overline{sg}(\psi(j) - j - 1);$$

$$\pi_\sigma(n) = \sum_{j=2}^{n} \overline{sg}(\sigma(j) - j - 1);$$

$$\pi_{\text{fr}}(n) = \sum_{j=2}^{n} \text{fr}\left(\frac{j}{(j-1)!}\right).$$

In [8], the author used the arithmetic function η, given for the natural number $n \geq 2$ with canonical form (1) by

$$\eta(n) = \sum_{i=1}^{n} \alpha_i . p_i,$$

for constructing the following new formula for π:

$$\pi_\eta(n) = \sum_{j=2}^{n} \overline{sg}(j - \eta(j)).$$

In 1987, the author introduced an arithmetic function with properties similar to the operation "differentiation" [1, 6]. For a natural number $n \geq 2$, with the canonical form (1), it has the form:

$$\delta(n) = \sum_{i=1}^{k} \alpha_i p_1^{\alpha_1} \ldots p_{i-1}^{\alpha_{i-1}} p_i^{\alpha_i - 1} p_{i+1}^{\alpha_{i+1}} \ldots p_k^{\alpha_k}.$$

Obviously, if p is a prime number, then from the definition it follows that

$$\delta(p) = 1.$$

In [9], the author used this function for constructing the following formula for π:

$$\pi_\delta(n) = \sum_{j=2}^{n} \left[\frac{1}{\delta(j)} \right].$$

If π is one of the representations $\pi_\varphi, \pi_\psi, \pi_\sigma, \pi_{fr}, \pi_\eta, \pi_\delta$, then for every natural number n:

$$p_n = \sum_{i=0}^{C(n)} \mathrm{sg}(n - \pi(i)),$$

where (see [13, 5.27, p. 90])

$$C(n) = \left[\frac{n^2 + 3n + 4}{4} \right].$$

Therefore, the formula for n-th prime number admits the forms

$$p_n = \sum_{i=0}^{C(n)} \mathrm{sg}\left(n - \sum_{j=2}^{i} \overline{\mathrm{sg}}(j - 1 - \varphi(j)) \right)$$

$$p_n = \sum_{i=0}^{C(n)} \mathrm{sg}\left(n - \sum_{j=2}^{i} \overline{\mathrm{sg}}(\psi(j) - j - 1) \right)$$

$$p_n = \sum_{i=0}^{C(n)} \mathrm{sg}\left(n - \sum_{j=2}^{i} \overline{\mathrm{sg}}(\sigma(j) - j - 1) \right)$$

$$p_n = \sum_{i=0}^{C(n)} \mathrm{sg}\left(n - \sum_{j=2}^{i} \mathrm{fr}\left(\frac{j}{(j-1)!} \right) \right)$$

$$p_n = \sum_{i=0}^{C(n)} \mathrm{sg}\left(n - \sum_{j=2}^{i} \overline{\mathrm{sg}}(j - \eta(j)) \right)$$

$$p_n = \sum_{i=0}^{C(n)} \mathrm{sg}\left(n - \sum_{j=2}^{i} \left[\frac{1}{\delta(j)} \right] \right),$$

where $\sum_{i=a}^{b} \bullet = 1$ for $a > b$.

In [20] Mladen Vassilev-Missana, continuing the idea from [3], proposed the following three formulas:

$$p_n = \sum_{j=0}^{C(n)} \left[\frac{1}{1 + \left[\frac{\pi(j)}{n} \right]} \right],$$

$$p_n = -2.\sum_{j=0}^{C(n)} \zeta\left(-2.\left[\frac{\pi(j)}{n} \right] \right),$$

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$$p_n = \sum_{j=0}^{C(n)} \frac{1}{\Gamma\left(1 - \left[\frac{\pi(j)}{n}\right]\right)},$$

where $\pi(j)$ can be evaluated by some one of the above formulas, ζ is Riemann's Zeta-function and Γ is Euler's Gamma-function.

3 New formulas for the n-th prime number[1]

First, we give the following new representations of formulas for π_φ, π_ψ, and π_σ:

$$\pi_\varphi(n) = \sum_{j=2}^{n} \left[\frac{\varphi(j)}{j-1}\right],$$

$$\pi_\psi(n) = \sum_{j=2}^{n} \left[\frac{j+1}{\psi(j)}\right],$$

$$\pi_\sigma(n) = \sum_{j=2}^{n} \left[\frac{j+1}{\sigma(j)}\right].$$

These formulas are equivaled to the previous ones, because if j is a prime number, then

$$\left[\frac{\varphi(j)}{j-1}\right] = \overline{\mathrm{sg}}(j-1-\varphi(j))$$

$$= \left[\frac{j+1}{\psi(j)}\right] = \overline{\mathrm{sg}}(\psi(j)-j-1)$$

$$= \left[\frac{j+1}{\sigma(j)}\right] = \overline{\mathrm{sg}}(\sigma(j)-j-1) = 1$$

and if j is a composite number, then

$$\left[\frac{\varphi(j)}{j-1}\right] = \overline{\mathrm{sg}}(j-1-\varphi(j))$$

$$= \left[\frac{j+1}{\psi(j)}\right] = \overline{\mathrm{sg}}(\psi(j)-j-1)$$

$$= \left[\frac{j+1}{\sigma(j)}\right] = \overline{\mathrm{sg}}(\sigma(j)-j-1) = 0.$$

Second, we introduce the following three well-known arithmetic functions (see, e.g., [14,15]), related to the canonical representation (1) of the natural number $n \geq 2$:

$$\tau(n) = \prod_{i=1}^{k}(1 + \alpha_i)$$

$$\omega(n) = k,$$

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[1]New, for the middle of 2018.

and

$$\Omega(n) = \sum_{i=1}^{k} \alpha_i.$$

Obviously, for every prime number p,

$$\omega(p) = \omega(p^2) = \omega(p^3) = \cdots = 1.$$

Let a natural number $n \geq 2$ is given by (1) and p_1 is the minimal prime numbers among $p_1, \ldots, p_k$. We introduce function $\overline{\omega}$ by

$$\overline{\omega}(n) = \alpha_1 \omega(n).$$

Then,

$$\overline{\omega}(n) = 1$$

if and only it n is a prime number.

Now, we can introduce the following three new formulas for the value of π:

$$\pi_\tau(n) = \sum_{j=2}^{n} \overline{\mathrm{sg}}(\tau(j) - 2) = \sum_{j=2}^{n} \left[\frac{1}{\tau(j) - 1} \right]$$

$$\pi_{\overline{\omega}}(n) = \sum_{j=2}^{n} \overline{\mathrm{sg}}(\overline{\omega}(j) - 1) = \sum_{j=2}^{n} \left[\frac{1}{\overline{\omega}(j)} \right],$$

and

$$\pi_\Omega(n) = \sum_{j=2}^{n} \overline{\mathrm{sg}}(\Omega(j) - 1) = \sum_{j=2}^{n} \left[\frac{1}{\Omega(j)} \right].$$

For the three cases, if j is a prime number, then

$$\left[\frac{1}{\tau(j) - 1} \right] = \overline{\mathrm{sg}}(\tau(j) - 2)$$

$$= \left[\frac{1}{\overline{\omega}(j)} \right] = \overline{\mathrm{sg}}(\overline{\omega}(j) - 1)$$

$$= \left[\frac{1}{\Omega(j)} \right] = \overline{\mathrm{sg}}(\Omega(j) - 1) = 1.$$

On the other hand, if j is a composite number, then $\tau(j) - 1 > 1$, $\overline{\omega}(j) > 1$ and $\Omega(j) > 1$, and

$$\left[\frac{1}{\tau(j) - 1} \right] = \overline{\mathrm{sg}}(\tau(j) - 2)$$

$$= \left[\frac{1}{\overline{\omega}(j)} \right] = \overline{\mathrm{sg}}(\overline{\omega}(j) - 1)$$

$$= \left[\frac{1}{\Omega(j)} \right] = \overline{\mathrm{sg}}(\Omega(j) - 1) = 0.$$

Therefore, the sum in the right-hand side of $\pi_\tau(n)$, $\pi_{\overline{\omega}}(n)$ and $\pi_\Omega(n)$ is equal to $\pi(n)$.

Finally, we can define function $\mathcal{P}$ over the natural number $n \geq 2$ with canonical representation (1), by:

$$\mathcal{P}(n) = \begin{cases} 1, & \text{if } n \text{ is a prime number} \\ 0, & \text{if } n \text{ is a composite number} \end{cases}.$$

Then

$$\pi_{\mathcal{P}}(n) = \sum_{j=2}^{n} \mathcal{P}(j).$$

Theorem 3.1. *For every natural number* n:

$$p_n = \sum_{i=0}^{C(n)} \text{sg}(n - \pi_{\tau}(i)), \tag{2}$$

$$p_n = \sum_{i=0}^{C(n)} \text{sg}(n - \pi_{\overline{\omega}}(i)), \tag{3}$$

$$p_n = \sum_{i=0}^{C(n)} \text{sg}(n - \pi_{\Omega}(i)), \tag{4}$$

$$p_n = \sum_{i=0}^{C(n)} \text{sg}(n - \pi_{\mathcal{P}}(i)). \tag{5}$$

Proof. Let us remind the fact (see [13]) that

$$p_n < C(n)$$

for every natural number n. Now, we can see that for a fixed natural number n, the expressions $n - \pi_{\tau}(i)$, $n - \pi_{\overline{\omega}}(i)$, $n - \pi_{\Omega}(i)$, and $n - \pi_{\mathcal{P}}(i)$ are monotonically decreasing with respect to i, and same is valid for $sg(n - \pi_{\tau}(i))$, $sg(n - \pi_{\overline{\omega}}(i))$, $sg(n - \pi_{\Omega}(i))$ and $sg(n - \pi_{\mathcal{P}}(i))$. When $i = 0, 1, \ldots, p_n - 1$, the numbers $n - \pi_{\tau}(i)$, $n - \pi_{\overline{\omega}}(i)$, $n - \pi_{\Omega}(i)$, and $sg(n - \pi_{\mathcal{P}}(i))$ are equal to 1, and when $i \geq p_n$, these numbers are equal to 0.

Therefore, the sums in (2), (3), (4), and (5) contain exactly p_n in number units, that proves the Theorem. $\square$

As a corrolary, we obtain the representations;

$$p_n = \sum_{i=0}^{C(n)} \text{sg}\left(n - \sum_{j=2}^{i} \left[\frac{1}{\tau(j) - 1}\right]\right)$$

$$p_n = \sum_{i=0}^{C(n)} \text{sg}\left(n - \sum_{j=2}^{i} \left[\frac{1}{\overline{\omega}(j)}\right]\right)$$

$$p_n = \sum_{i=0}^{C(n)} \text{sg}\left(n - \sum_{j=2}^{i} \left[\frac{1}{\Omega(j)}\right]\right),$$

$$p_n = \sum_{i=0}^{C(n)} \text{sg}\left(n - \sum_{j=2}^{i} \left[\frac{1}{\Omega(j)}\right]\right).$$

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At the end, we mention that the following **Open Problem** is interesting: Which formula for p_n has the minimal computational compexity?

4 Instead of Conclusion: Four new formulas for the n-th prime number

During the last 25 years, the author introduced some new arithmetic functions that also can be used for determination of the n-th prime number. They are (using representation (1) for n):

- Irrational factor [2, 19]:

$$IF(n) = \prod_{i=1}^{k} p_i^{1/\alpha_i},$$

- Converse factor [4, 19]:

$$CF(n) = \prod_{i=1}^{k} \alpha_i^{p_i},$$

- Restrictive factor [5, 19]:

$$RF(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1},$$

- a function, in some sense dual to σ [7]

$$\chi(n) = \prod_{i=1}^{k} (p_i^{\alpha_i} - p_i^{\alpha_i - 1} + \cdots + (-1)^{\alpha_i - 1} p_i + (-1)^{\alpha_i})$$

For these functions we obtain:

$$\pi_{IF}(n) = \sum_{i=2}^{n} \left[\frac{1}{\omega(n) IF(n)} \right],$$

$$\pi_{CF}(n) = \sum_{i=2}^{n} \left[\frac{1}{\omega(n) CF(n)} \right],$$

$$\pi_{RF}(n) = \sum_{i=2}^{n} \left[\frac{1}{\omega(n) RF(n)} \right],$$

$$\pi_{\chi}(n) = \sum_{j=2}^{n} \overline{\mathrm{sg}}(j - 1 - \chi(j))$$

and

$$p_n = \sum_{i=0}^{C(n)} \mathrm{sg}(n - \pi_{IF}(i)) = \sum_{i=0}^{C(n)} \mathrm{sg}(n - \pi_{CF}(i)) = \sum_{i=0}^{C(n)} \mathrm{sg}(n - \pi_{RF}(i)) = \sum_{i=0}^{C(n)} \mathrm{sg}(n - \pi_{\chi}(i)).$$

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