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Pauli–Fibonacci quaternions

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Abstract: The aim of this work is to consider the Pauli–Fibonacci quaternions and to present some properties involving this sequence, including the Binet's formula and generating functions. Furthermore, the Honsberger identity, the generating function, d'Ocagne's identity, Cassini's identity, Catalan's identity for these quaternions are given. The matrix representations for Pauli–Fibonacci quaternions are introduced.

Keywords: Pauli matrix, Pauli quaternion, Fibonacci number, Fibonacci quaternion, Pauli–Fibonacci quaternion.
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1 Introduction

The real quaternions were first described by Irish mathematician William Rowan Hamilton in 1843. The real quaternions constitute an extension of complex numbers into a four-dimensional space and can be considered as four-dimensional vectors, in the same way that complex numbers are considered as two-dimensional vectors.

In [11], Hamilton introduced the set of real quaternions which can be represented as

$$H = \{\, q = q_0 + i\, q_1 + j\, q_2 + k\, q_3 \,|\, q_s \in \mathbb{R}\,, s = 0, 1, 2, 3 \,\} \tag{1}$$

where

$$i^2 = j^2 = k^2 = -1\,,\;\; i\, j = -j\, i = k\,, \quad j\, k = -k\, j = i\,, \quad k\, i = -i\, k = j\,.$$

Quaternions have received widespread attention for their potential use in a new formulation of quantum mechanics and quantum field theory [1]. Horadam [12, 13] defined complex Fibonacci and Lucas quaternions as follows

$$Q_n = F_n + F_{n+1}\, i + F_{n+2}\, j + F_{n+3}\, k \tag{2}$$

and

$$K_n = L_n + L_{n+1}\, i + L_{n+2}\, j + L_{n+3}\, k \tag{3}$$

where F_n and L_n denote the n-th Fibonacci and Lucas numbers, respectively. Also, the imaginary quaternion units $i,\ j,\ k$ have the following rules

$$i^2 = j^2 = k^2 = -1\,,\ \ i\,j = -j\,i = k\,,\quad j\,k = -k\,j = i\,,\quad k\,i = -i\,k = j$$

There are several studies on different quaternions and their generalizations, for example [2, 9, 10, 14, 15, 19, 22, 25].

The Pauli matrices have applications in different areas of mathematics and mathematical physics [4, 5, 7, 8, 16, 18]. The work on the the Pauli matrices can be found in [4, 5, 16]. The Pauli matrices are Hermitian and unitary which are elements of a set of three 2×2 complex matrices as follows:

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

whose multiplication rules are

$$\begin{aligned} \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1 \qquad & \sigma_1\,\sigma_2 = -\sigma_2\,\sigma_1 = i\,\sigma_3\,, \\ \sigma_2\,\sigma_3 = -\sigma_3\,\sigma_2 = i\,\sigma_1, \quad & \sigma_3\,\sigma_1 = -\sigma_1\,\sigma_3 = i\,\sigma_2. \end{aligned} \tag{5}$$

The famous physicist Wolfgang Pauli has introduced the Pauli matrices [5, 16]. The Pauli quaternions are defined by the basis $\{1,\, i\,\sigma_1\,, i\,\sigma_2\,, i\,\sigma_3\,\}$ [12]. This set of base is isomorphic to quaternions $\mathbb{H}$. In [16], the isomorphism from $\mathbb{H}$ to this set is given by the following map which are reversed signs for the Pauli matrices:

$$1 \to I\,,\ i \to -i\,\sigma_1\,,\ j \to -i\,\sigma_2\,,\ k \to -i\,\sigma_3\,.$$

The Hamilton multiplication rules differ from the Pauli matrix rules only by a factor of i. It is possible to formulate special relativity with Hamilton quaternions having complex coefficients (called biquaternions) [11].

In quantum mechanics, they occur in the Pauli equation which takes into account the interaction of the spin of a particle with an external electromagnetic field. It turns out that the formulae of general relativity are simpler with the Pauli quaternions [21]. There is also a very interesting relation between the Pauli quaternions and three-dimensional Clifford algebra [6].

Hermitian operators represent observables in quantum mechanics, so the Pauli matrices span the space of observables of the two-dimensional complex Hilbert space. In the context of Pauli's work, σk represents the observable corresponding to spin along the k-th coordinate axis in three-dimensional Euclidean space [3, 7].

The Pauli matrices (after multiplication by i to make them anti-Hermitian) also generate transformations in the sense of Lie algebras: the matrices $i\,\sigma_1\,, i\,\sigma_2\,, i\,\sigma_3$ form a basis for the real Lie algebra, which exponentiates to the special unitary group $SU(2)$. The algebra generated by the three matrices $\sigma_1\,, \sigma_2\,, \sigma_3$ is isomorphic to the Clifford algebra and the algebra generated by $i\,\sigma_1\,, i\,\sigma_2\,, i\,\sigma_3$ is isomorphic to the quaternions.

The Pauli matrices are closely related to two-dimensional representations of $SO(3)$ and $SU(2)$ groups ($SO(3)$ and $SU(2)$ groups are isomorphic). And they are used in representing rotation. A necessary and sufficient condition for a rotation to be representative is that it satisfies the Pauli matrices. But it can be easily verified that only two-dimensional representations of $SO(3)$ satisfy this property [3].

In 2017, Kim [16] defined the Pauli quaternions $\mathbb{H}_P$ and De Moivre's formula of these quaternions, as follows

$$q = x_0\,1 + x_1\,\sigma_1 + x_2\,\sigma_2 + x_3\,\sigma_3 \tag{6}$$

Also, the quaternion units have the rules (5).
The Pauli-quaternion product can be written as

$$q.p = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_0 & -ix_3 & ix_2 \\ x_2 & ix_3 & x_0 & -ix_1 \\ x_3 & -ix_2 & ix_1 & x_0 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

The base elements of the Pauli-quaternions satisfy the following commutative multiplication scheme (Table 1).

x	1	σ_1	σ_2	σ_3
1	1	σ_1	σ_2	σ_3
σ_1	σ_1	1	$i\sigma_3$	$-i\sigma_2$
σ_2	σ_2	$-i\sigma_3$	1	$i\sigma_1$
σ_3	σ_3	$i\sigma_2$	$-i\sigma_1$	1

Table 1. Multiplication scheme of the Pauli-quaternionic units

The conjugate of the Pauli-quaternion [16] as follows:

$$\begin{aligned} q &= x_0\,1 + x_1\,\sigma_1 + x_2\,\sigma_2 + x_3\,\sigma_3, \\ q^* &= x_0\,1 - x_1\,\sigma_1 - x_2\,\sigma_2 - x_3\,\sigma_3. \end{aligned} \tag{7}$$

Moreover, $q\,q^* = q^*\,q = (x_0^2 - x_1^2 - x_2^2 + x_3^2)\,1$. Also, the norm of Pauli-quaternion is defined as

$$N_q = \|q \times q^*\| = \sqrt{|x_0^2 - x_1^2 - x_2^2 + x_3^2|}. \tag{8}$$

If $N_p = 1$, then $p \in \mathbb{H}_p^* = \mathbb{H}_p - E$ is called unit Pauli quaternions. Also, spacelike and timelike Pauli quaternions have multiplicative inverse, denoted by p^{-1}, where

$$E = \{x_0 1 + x_1 \sigma 1 + x_2 \sigma 2 + x_3 \sigma 3 | \, x_0^2 = x_1^2 + x_2^2 + x_3^2\}$$

and their property $p\,p^{-1} = p^{-1}\,p = 1$. On the other hand, lightlike Pauli quaternions have no inverses [20].

In this paper, the Pauli–Fibonacci quaternions will be defined. In addition, the Honsberger identity, the d'Ocagne's identity, the generating function, Binet's formula, Cassini's identity, Catalan's identity for these quaternions are given.

2 The Pauli–Fibonacci quaternions

The Pauli–Fibonacci and Pauli–Lucas quaternions can be defined by the basis $\{1, \, i\,\sigma_1\,, i\,\sigma_2\,, i\,\sigma_3\,\}$, where $i\sigma_1, i\sigma_2$ and $i\sigma_3$ satisfy the conditions (5) as follows

$$Q_p F_n = F_n + F_{n+1}\,\sigma_1 + F_{n+2}\,\sigma_2 + F_{n+3}\,\sigma_3 \tag{9}$$

and

$$Q_p L_n = L_n + L_{n+1}\,\sigma_1 + L_{n+2}\,\sigma_2 + L_{n+3}\,\sigma_3 \tag{10}$$

The addition, substraction and multiplication by real scalars of two Pauli–Fibonacci quaternions gives the Pauli–Fibonacci quaternion. Then, the addition and subtraction of the Pauli–Fibonacci quaternions are defined by

$$\begin{aligned} Q_p F_n \pm Q_p F_m = \; & (F_n \pm F_m) + (F_{n+1} \pm F_{m+1})\,.\sigma_1 \\ & + (F_{n+2} \pm F_{m+2})\,.\sigma_2 + (F_{n+3} \pm F_{m+3})\,.\sigma_3. \end{aligned} \tag{11}$$

The multiplication of a Pauli–Fibonacci quaternion by the real scalar λ is defined as

$$\lambda\, Q_p F_n = \lambda\, F_n + \lambda\, F_{n+1}\,.\sigma_1 + \lambda\, F_{n+2}\,.\sigma_2 + \lambda\; F_{n+3}\,.\sigma_3. \tag{12}$$

By using (Table 1) the multiplication of two Pauli–Fibonacci quaternions is defined by

$$\begin{aligned} Q_p F_n \times Q_p F_m = \; & (F_n\, F_m + F_{n+1}\, F_{m+1} + F_{n+2}\, F_{m+2} + F_{n+3}\, F_{m+3}).1 \\ & + (F_n\, F_{m+1} + F_{n+1}\, F_m + i\,(-1)^m\, F_{n-m})\,.\sigma_1 \\ & + (F_n\, F_{m+2} + F_{n+2}\, F_m + i\,(-1)^m\, F_{n-m})\,.\sigma_2 \\ & + (F_n\, F_{m+3} + F_{n+3}\, F_m + i\,(-1)^{m+1}\, F_{n-m})\,.\sigma_3 \\ = \; & Q_p F_m \times Q_p F_n. \end{aligned} \tag{13}$$

The scalar and the vector part of $Q_p F_n$ which is the n-th term of the Pauli–Fibonacci quaternion with $(Q_p F_n)$ are denoted by

$$S_{Q_p F_n} = F_n \;\text{ and }\; V_{Q_p F_n} = F_{n+1}\,\sigma_1 + F_{n+2}\,\sigma_2 + F_{n+3}\,\sigma_3. \tag{14}$$

Thus, the Pauli–Fibonacci quaternion Q_pF_n is given by $Q_pF_n = S_{Q_pF_n} + V_{Q_pF_n}$. Then, relation (13) is defined by

$$\begin{aligned} Q_pF_n \times Q_pF_m = \; & S_{Q_pF_n} S_{Q_pF_m} + \langle V_{Q_pF_n}, V_{Q_pF_m} \rangle + S_{Q_pF_n} V_{Q_pF_m} \\ & + S_{Q_pF_m} V_{Q_pF_n} + V_{Q_pF_n} \wedge V_{Q_pF_m} . \end{aligned} \tag{15}$$

Also, the Pauli–Fibonacci quaternion product may be obtained as follows:

$$Q_pF_n \times Q_pF_m = \begin{pmatrix} F_n & F_{n+1} & F_{n+2} & F_{n+3} \\ F_{n+1} & F_n & -iF_{n+3} & iF_{n+2} \\ F_{n+2} & iF_{n+3} & F_n & -iF_{n+1} \\ F_{n+3} & -iF_{n+2} & iF_{n+1} & F_n \end{pmatrix} \begin{pmatrix} F_m \\ F_{m+1} \\ F_{m+2} \\ F_{m+3} \end{pmatrix}.$$

The conjugate of the Pauli–Fibonacci quaternion Q_pF_n is denoted by $\overline{Q_pF_n}$ and it is

$$\overline{Q_pF_n} = F_n - F_{n+1}\sigma_1 - F_{n+2}\sigma_2 - F_{n+3}\sigma_3. \tag{16}$$

The norm of Q_pF_n is defined as follows

$$\|Q_pF_n\|^2 = Q_pF_n \, \overline{Q_pF_n} = |F_n^2 - F_{n+1}^2 - F_{n+2}^2 - F_{n+3}^2|. \tag{17}$$

In the following theorem, some properties related to Pauli–Fibonacci quaternions are given.

Theorem 1. *Let F_n and Q_pF_n be the n-th terms of Fibonacci sequence (F_n) and Pauli–Fibonacci quaternion (Q_pF_n), respectively. In this case, for $n \geq 1$ we can give the following relations:*

$$Q_pF_{n+1} = Q_pF_n + Q_pF_{n-1}, \tag{18}$$

$$Q_pF_{n+1} + Q_pF_{n-1} = Q_pL_n, \tag{19}$$

$$Q_pF_{n+2} - Q_pF_{n-2} = Q_pL_n, \tag{20}$$

$$Q_pF_n - Q_pF_{n+1}\sigma_1 - Q_pF_{n+2}\sigma_2 - Q_pF_{n+3}\sigma_3 = F_n - F_{n+2} - F_{n+4} - F_{n+6}. \tag{21}$$

Using (9) and (10) proof can easily be done.

Theorem 2 (Honsberger identity)**.** *For $n, m \geq 0$ the Honsberger identity for the Pauli–Fibonacci quaternions Q_pF_n and Q_pF_m is given by*

$$Q_pF_n \, Q_pF_m + Q_pF_{n+1} \, Q_pF_{m+1} = 2\, Q_pF_{n+m+1} + 9\, F_{n+m+1} + 5\, F_{n+m+2} \tag{22}$$

Proof. By using (9) we get,

$$\begin{aligned} Q_pF_n \, Q_pF_m + Q_pF_{n+1} \, Q_pF_{m+1} = \; & (F_{n+m+1} + F_{n+m+3} + F_{n+m+5} + F_{n+m+7}) \\ & +2\,(F_{n+m+2}\,\sigma_1 + F_{n+m+3}\,\sigma_2 + F_{n+m+4}\,\sigma_3) \\ = \; & 2\, Q_pF_{n+m+1} + 9\, F_{n+m+1} + 5\, F_{n+m+2}. \end{aligned}$$

where the identity $F_nF_m + F_{n+1}F_{m+1} = F_{n+m+1}$ is used. □

Theorem 3 (Generating function)**.** *Let Q_pF_n be the Pauli–Fibonacci quaternion. For the generating function for these quaternions is as follows:*

$$g_{Q_pF_n}(t) = \sum_{s=0}^{n} Q_pF_n\, t^n = \frac{Q_pF_0 + (Q_pF_1 - Q_pF_0)\, t}{1 - t - t^2} \tag{23}$$

Proof. Using the definition of generating function, we obtain

$$g_{Q_pF_n}(t) = Q_pF_0 + Q_pF_1\, t + \cdots + Q_pF_n\, t^n + \cdots . \tag{24}$$

Multiplying by $(1 - t - t^2)$ both sides of (24) and using (18), we have

$$(1 - t - t^2)\, g_{Q_pF_n}(t) = Q_pF_0 + (Q_pF_1 - Q_pF_0)\, t .$$

Thus, the proof is completed. □

Theorem 4 (Binet's formula)**.** *Let Q_pF_n be the Pauli–Fibonacci quaternion. For $n \geq 1$, Binet's formula for these quaternions is as follows:*

$$Q_pF_n = \frac{1}{\alpha - \beta}\left(\hat{\alpha}\ \alpha^n - \hat{\beta}\ \beta^n\right) \tag{25}$$

where

$$\hat{\alpha} = 1 + \alpha\,\sigma_1 + \alpha^2\,\sigma_2 + \alpha^3\,\sigma_3, \quad \alpha = \frac{1+\sqrt{5}}{2}$$

and

$$\hat{\beta} = 1 + \beta\,\sigma_1 + \beta^2\,\sigma_2 + \beta^3\,\sigma_3, \quad \beta = \frac{1-\sqrt{5}}{2}.$$

Proof. Using (9) and Binet's formula of the Fibonacci quaternion [9], the proof is easily seen.

$$\begin{aligned}
Q_pF_n =&\ F_n\, 1 + F_{n+1}\,.\sigma_1 + F_{n+2}\,.\sigma_2 + F_{n+3}\,.\sigma_3 \\
=&\ (\frac{\alpha^n - \beta^n}{\alpha - \beta})\, 1 + (\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta})\,\sigma_1 + \varepsilon\,(\frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta})\,\sigma_2 + (\frac{\alpha^{n+3} - \beta^{n+3}}{\alpha - \beta})\,\sigma_3 \\
=&\ \frac{\alpha^n\,(1 + \alpha\,\sigma_1 + \alpha^2\,\sigma_2 + \alpha^3\,\sigma_3) - \beta^n\,(1 + \beta\,\sigma_1 + \beta^2\,\sigma_2 + \beta^3\,\sigma_3)}{\alpha - \beta} \\
=&\ \frac{1}{\sqrt{5}}\left(\hat{\alpha}\,\alpha^n - \hat{\beta}\,\beta^n\right)
\end{aligned}$$

where $\hat{\alpha} = 1 + \alpha\,\sigma_1 + \alpha^2\,\sigma_2 + \alpha^3\,\sigma_3, \quad \hat{\beta} = 1 + \beta\,\sigma_1 + \beta^2\,\sigma_2 + \beta^3\,\sigma_3.$ □

Theorem 5 (d'Ocagne's identity)**.** *For $n, m \geq 0$ the d'Ocagne's identity for the Pauli–Fibonacci quaternions Q_pF_n and Q_pF_m is given by*

$$\begin{aligned}
Q_pF_m\, Q_pF_{n+1} - Q_pF_{m+1}\, Q_pF_n =&\ (-1)^n\, F_{m-n}\,(\,\sigma_1 + 3\,\sigma_2 + 4\,\sigma_3\,) \\
&-i\, L_{m-n}\,(\sigma_1\ + \sigma_2 - \sigma_3\,).
\end{aligned} \tag{26}$$

Proof. By using (9) we get,

$$\begin{aligned} Q_pF_m\, Q_pF_{n+1} - Q_pF_{m+1}\, Q_pF_n = \ & [\,(-1)^n\, F_{m-n} + i\,(\,(-1)^{n+1}\, L_{m-n})\,]\,\sigma_1 \\ & +[\,(-1)^n\,(3\,F_{m-n} + i\,(\,(-1)^{n+1}\, L_{m-n})\,]\,\sigma_2 \\ & +[\,(-1)^n\,(4\,F_{m-n} + i\,(\,(-1)^{n}\, L_{m-n})\,]\,\sigma_3. \end{aligned}$$

where the identities $F_nF_m + F_{n+1}F_{m+1} = F_{n+m+1}\,, F_{n+2} - F_{n-2} = 3\,F_n,\ F_{n+3} - F_{n-3} = 4\,F_n$ and $F_{n+1} - F_{n-1} = L_n$ are used [17, 23, 24]. □

Theorem 6 (Cassini's identity)**.** *Let Q_pF_n be the Pauli–Fibonacci quaternion. For $n \geq 1$, Cassini's identity for Q_pF_n is as follows:*

$$Q_pF_n^2 - Q_pF_{n+1}\, Q_pF_{n-1} = (-1)^{n+1}\,[\,(1-i)\,\sigma_1 + (3-i)\,\sigma_2 + (4+i)\,\sigma_3\,]. \tag{27}$$

Proof. By using (9) we get

$$\begin{aligned} (Q_pF_n)^2 - Q_pF_{n+1}\, Q_pF_{n-1} = \ & [\,(F_n^2 - F_{n+1}F_{n-1}) + (F_{n+1}^2 - F_{n+2}F_n) \\ & +(F_{n+2}^2 - F_{n+3}F_{n+1}) + (F_{n+3}^2 - F_{n+4}F_{n+2})\,] \\ & +[\,(F_{n+1}F_n - F_{n+2}F_{n-1}) - i\,(-1)^{n+1}\,]\,\sigma_1 \\ & +[\,(F_{n+2}F_n - F_{n+3}F_{n-1}) \\ & +(F_nF_{n+2} - F_{n+1}F_{n+1}) - i\,(-1)^{n-1}\,]\,\sigma_2 \\ & +[\,(F_{n+3}F_n - F_{n+4}F_{n-1}) \\ & +(F_nF_{n+3} - F_{n+1}F_{n+2}) - i\,(-1)^{n}\,]\,\sigma_3 \\ = \ & (-1)^{n+1}\,[\,(1-i)\,\sigma_1 + (3-i)\,\sigma_2 + (4+i)\,\sigma_3\,]. \end{aligned}$$

where the identity of the Fibonacci numbers $F_mF_{n+1} - F_{m+1}F_n = (-1)^nF_{m-n}$ is used [17, 23, 24]. □

Theorem 7 (Catalan's identity)**.** *Let Q_pF_n be the Pauli–Fibonacci quaternion. For $n \geq 1$, Catalan's identity for Q_pF_n is as follows:*

$$Q_pF_n^2 - Q_pF_{n+r}\, Q_pF_{n-r} = (-1)^{n-r}\, F_r\,[\,((1-i)\,\sigma_1 + (3-i)\,\sigma_2 + (4+i)\,\sigma_3)\,]. \tag{28}$$

Proof. By using (9) we get

$$\begin{aligned} Q_pF_n^2 - Q_pF_{n+r}\, Q_pF_{n-r} = \ & [\,(F_n^2 - F_{n+r}\,F_{n-r}) - (F_{n+1}^2 - F_{n+r+1}\,F_{n-r+1}) \\ & \quad +(F_{n+2}^2 - F_{n+r+2}\,F_{n-r+2}) \\ & \quad +(F_{n+3}^2 - F_{n+r+3}\,F_{n-r+3})\,] \\ & +[\,(F_n\,F_{n+1} - F_{n+r}\,F_{n-r+1}) \\ & \quad +(F_{n+1}\,F_n - F_{n+r+1}\,F_{n-r}) + i\,(-1)^{n-r+1}\,]\,\sigma_1 \\ & +[\,(F_n\,F_{n+2} - F_{n+r}\,F_{n-r+2}) \\ & \quad -(F_{n+2}\,F_n - F_{n+r+2}\,F_{n-r}) + i\,(-1)^{n-r+1}\,]\,\sigma_2 \\ & +[\,(F_n\,F_{n+3} - F_{n+r}\,F_{n-r+3}) \\ & \quad +(F_{n+3}\,F_n - F_{n+r+3}\,F_{n-r}) - i\,(-1)^{n-r+1}\,]\,\sigma_3 \\ = \ & (-1)^{n-r}\,F_r\,[\,(1-i)\,\sigma_1 + (3-i)\,\sigma_2 + (4+i)\,\sigma_3\,]. \end{aligned}$$

where the identities of the Fibonacci numbers $F_n^2 - F_{n+r}\,F_{n-r} = (-1)^{n-r}\,F_r^2$ and $F_mF_n - F_{m+r}F_{n-r} = (-1)^{n-r}F_{m+r-n}\,F_r$ are used [17, 23, 24]. □

3 Representations of Pauli–Fibonacci quaternions

In this section R-linear transformations are introduced, representing left and right multiplication in $\mathbb{H}_pF_n$ by using the De Moivre's formula for a corresponding matrix representation. Let χ be a Pauli–Fibonacci quaternion, then, $\varphi_{L_p}(\chi)$ and $\varphi_{R_p}(\chi)$ defined as follows: for $\chi \in \mathbb{H}_pF_n$,

$$\begin{aligned}\varphi_{L_p} :& \mathbb{H}_pF_n \to \mathbb{H}_pF_n \\ &\chi \to \varphi_{L_p}(\chi) = A_{\varphi_{L_p}}\,\chi\end{aligned}$$

$$A_{\varphi_{L_p}} = \begin{pmatrix} F_n & F_{n+1} & F_{n+2} & F_{n+3} \\ F_{n+1} & F_n & -i\,F_{n+3} & i\,F_{n+2} \\ F_{n+2} & i\,F_{n+3} & F_n & -i\,F_{n+1} \\ F_{n+3} & -i\,F_{n+2} & i\,F_{n+1} & F_n \end{pmatrix}$$

and

$$\begin{aligned}\varphi_{R_p} :& \mathbb{H}_pF_n \to \mathbb{H}_pF_n \\ &\chi \to \varphi_{R_p}(\chi) = \chi\,A_{\varphi_{R_p}}\end{aligned}$$

$$A_{\varphi_{R_p}} = \begin{pmatrix} F_n & F_{n+1} & F_{n+2} & F_{n+3} \\ F_{n+1} & F_n & i\,F_{n+3} & -i\,F_{n+2} \\ F_{n+2} & -i\,F_{n+3} & F_n & i\,F_{n+1} \\ F_{n+3} & i\,F_{n+2} & -i\,F_{n+1} & F_n \end{pmatrix},$$

respectively. For any $Q_pF_m,\ Q_pF_n \in \mathbb{H}_pF_n$ and $\lambda \in \mathbb{R}$, the following properties hold:

$$\begin{aligned}\varphi_{L_p}(Q_pF_m + Q_pF_n) &= \varphi_{L_p}(Q_pF_m) + \varphi_{L_p}(Q_pF_n) \\ \varphi_{L_p}(\lambda\,Q_pF_n) &= \lambda\,\varphi_{L_p}(Q_pF_n) \\ \varphi_{R_p}(Q_pF_m + Q_pF_n) &= \varphi_{R_p}(Q_pF_m) + \varphi_{R_p}(Q_pF_n) \\ \varphi_{R_p}(\lambda\,Q_pF_n) &= \lambda\,\varphi_{R_p}(Q_pF_n), \\ \varphi_{L_p}(Q_pF_n)\,\varphi_{R_p}(Q_pF_n) &= \varphi_{R_p}(Q_pF_n)\,\varphi_{L_p}(Q_pF_n).\end{aligned}$$

Furthermore, linear mappings of $\psi_{L_p}(Q_pF_n)$ and $\psi_{L_p}(Q_pF_n)$ defined as

$$\psi_{L_p} :(\mathbb{H}_pF_n, +, .) \to (M_{(4,R)}, \oplus, \otimes)$$

$$\psi_{L_p}(F_n + F_{n+1}\,\sigma_1 + F_{n+2}\,\sigma_2 + F_{n+3}\,\sigma_3) \to \begin{pmatrix} F_n & F_{n+1} & F_{n+2} & F_{n+3} \\ F_{n+1} & F_n & -i\,F_{n+3} & i\,F_{n+2} \\ F_{n+2} & i\,F_{n+3} & F_n & -i\,F_{n+1} \\ F_{n+3} & -i\,F_{n+2} & i\,F_{n+1} & F_n \end{pmatrix}$$

and

$$\psi_{R_p} :(\mathbb{H}_pF_n, +, .) \to (M_{(4,R)}, \oplus, \otimes)$$

$$\psi_{R_p}(F_n + F_{n+1}\,\sigma_1 + F_{n+2}\,\sigma_2 + F_{n+3}\,\sigma_3) \to \begin{pmatrix} F_n & F_{n+1} & F_{n+2} & F_{n+3} \\ F_{n+1} & F_n & i\,F_{n+3} & -i\,F_{n+2} \\ F_{n+2} & -i\,F_{n+3} & F_n & i\,F_{n+1} \\ F_{n+3} & i\,F_{n+2} & -i\,F_{n+1} & F_n \end{pmatrix}$$

are one-to-one and onto. Also, the following properties hold:

$$\psi_{L_p}((Q_pF_m)(Q_pF_n)) = \psi_{L_p}(Q_pF_m)\,\psi_{L_p}(Q_pF_n)$$

and

$$\psi_{R_p}((Q_pF_m)(Q_pF_n)) = \psi_{R_p}(Q_pF_m)\,\psi_{R_p}(Q_pF_n).$$

Therefore, the mappings ψ_{L_p} and ψ_{R_p} are isomorphisms [5, 16].

4 Conclusion

In this paper, algebraic and analytic properties of Pauli–Fibonacci quaternions are investigated. Matrix representations of these quaternions are also given. I hope that these results will be important in applied mathematics, quantum physics, Lie groups and kinematics.

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