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On Ramanujan type identities and Cardano formula

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Abstract: In this paper we will prove some Ramanujan type identities such as

$$\begin{aligned}
&\sqrt[3]{\sin\left(\frac{\pi}{9}\right)}+\sqrt[3]{\sin\left(\frac{2\pi}{9}\right)}+\sqrt[3]{\sin\left(\frac{14\pi}{9}\right)}\\
&\qquad=\left(-\frac{\sqrt[18]{3}}{2}\right)\left(\sqrt[3]{6+3\left(\sqrt[3]{6-3\sqrt[3]{9}}+\sqrt[3]{3-3\sqrt[3]{9}}\right)}\right),\\
&\sqrt[3]{\tan\left(\frac{\pi}{9}\right)}+\sqrt[3]{\tan\left(\frac{4\pi}{9}\right)}+\sqrt[3]{\tan\left(\frac{7\pi}{9}\right)}\\
&\qquad=\left(-\sqrt[18]{3}\right)\left(\sqrt[3]{-3\sqrt[3]{3}+6+3(\sqrt[3]{21-3(3\sqrt[3]{3}-\sqrt[3]{9})}-\sqrt[3]{3+3(3\sqrt[3]{3}+\sqrt[3]{9})})}\right).
\end{aligned}$$

Keywords: Ramanujan type identity, Trigonometric function, Cubic equation, Radicals.
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1 Introduction

Ramanujan proved the following most well-known identities in his notebook [2]:

$$\sqrt[3]{\cos\left(\frac{2\pi}{7}\right)}+\sqrt[3]{\cos\left(\frac{4\pi}{7}\right)}+\sqrt[3]{\cos\left(\frac{8\pi}{7}\right)}=\sqrt[3]{\frac{5-3\sqrt[3]{7}}{2}}, \tag{1}$$

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$$\sqrt[3]{\cos\left(\frac{2\pi}{9}\right)} + \sqrt[3]{\cos\left(\frac{4\pi}{9}\right)} + \sqrt[3]{\cos\left(\frac{8\pi}{9}\right)} = -\sqrt[3]{\frac{6-3\sqrt[3]{9}}{2}}. \tag{2}$$

It is natural to ask if there are similar identities for sine and tangent functions. It turns out that there are similar identities for sine and tangent functions at angles $\pi/7$'s [6]. Since we could not find such identities for sine and tangent functions at angles $\pi/9$'s in the literature so we decide to find similar identities. This is the motivation of this research.

However, after reviewing, it turned out that some type of these inequalities have already been considered in papers [7, 8].

The purpose of this paper is to prove identities which are similar to identities (1), (2) for sine and tangent functions. The identities are much more complicated. Our main results are the following.

Theorem 1.1.

$$\sqrt[3]{\sin\left(\frac{\pi}{9}\right)} + \sqrt[3]{\sin\left(\frac{2\pi}{9}\right)} + \sqrt[3]{\sin\left(\frac{14\pi}{9}\right)} = \left(-\frac{\sqrt[18]{3}}{2}\right)\left(\sqrt[3]{6+3\left(\sqrt[3]{6-3\sqrt[3]{9}}+\sqrt[3]{3-3\sqrt[3]{9}}\right)}\right), \tag{3}$$

$$\frac{1}{\sqrt[3]{\sin(\frac{\pi}{9})}} + \frac{1}{\sqrt[3]{\sin(\frac{2\pi}{9})}} + \frac{1}{\sqrt[3]{\sin(\frac{14\pi}{9})}} = \left(-\frac{2}{\sqrt[18]{3}}\right)\left(\sqrt[3]{-\sqrt[3]{9}+6+3\left(\sqrt[3]{6-3\sqrt[3]{9}}+\sqrt[3]{3-3\sqrt[3]{9}}\right)}\right). \tag{4}$$

Theorem 1.2.

$$\sqrt[3]{\tan\left(\frac{\pi}{9}\right)} + \sqrt[3]{\tan\left(\frac{4\pi}{9}\right)} + \sqrt[3]{\tan\left(\frac{7\pi}{9}\right)} = \left(-\sqrt[18]{3}\right)\left(\sqrt[3]{-3\sqrt[3]{3}+6+3(\sqrt[3]{21-3(3\sqrt[3]{3}-\sqrt[3]{9})}-\sqrt[3]{3+3(3\sqrt[3]{3}+\sqrt[3]{9})})}\right), \tag{5}$$

$$\frac{1}{\sqrt[3]{\tan\left(\frac{\pi}{9}\right)}} + \frac{1}{\sqrt[3]{\tan\left(\frac{4\pi}{9}\right)}} + \frac{1}{\sqrt[3]{\tan\left(\frac{7\pi}{9}\right)}} = \left(-\frac{1}{\sqrt[18]{3}}\right)\left(\sqrt[3]{-\sqrt[3]{9})+6+3(\sqrt[3]{21-3(3\sqrt[3]{3}-\sqrt[3]{9})}-\sqrt[3]{3+3(3\sqrt[3]{3}+\sqrt[3]{9})})}\right). \tag{6}$$

Theorem 1.3.

$$\sqrt[3]{\sin\left(\frac{2\pi}{7}\right)} + \sqrt[3]{\sin\left(\frac{4\pi}{7}\right)} + \sqrt[3]{\sin\left(\frac{8\pi}{7}\right)} = \left(-\sqrt[18]{\frac{7}{64}}\right)\left(\sqrt[3]{-\sqrt[3]{7}+6+3\left(\sqrt[3]{5-3\sqrt[3]{7}}+\sqrt[3]{4-3\sqrt[3]{7}}\right)}\right), \tag{7}$$

$$\frac{1}{\sqrt[3]{\sin(\frac{2\pi}{7})}}+\frac{1}{\sqrt[3]{\sin(\frac{4\pi}{7})}}+\frac{1}{\sqrt[3]{\sin(\frac{8\pi}{7})}}=$$

$$\left(-\sqrt[18]{\frac{64}{7}}\right)\left(\sqrt[3]{6+3\left(\sqrt[3]{5-3\sqrt[3]{7}}+\sqrt[3]{4-3\sqrt[3]{7}}\right)}\right). \quad (8)$$

Theorem 1.4.

$$\sqrt[3]{\tan\left(\frac{2\pi}{7}\right)}+\sqrt[3]{\tan\left(\frac{4\pi}{7}\right)}+\sqrt[3]{\tan\left(\frac{8\pi}{7}\right)}=$$

$$\left(\sqrt[18]{7}\right)\left(\sqrt[3]{-\sqrt[3]{7}+6+3\left(\sqrt[3]{3\sqrt[3]{7}-3\sqrt[3]{49}+5}+\sqrt[3]{3\sqrt[3]{7}-3\sqrt[3]{49}-3}\right)}\right), \quad (9)$$

$$\frac{1}{\sqrt[3]{\tan\left(\frac{2\pi}{7}\right)}}+\frac{1}{\sqrt[3]{\tan\left(\frac{4\pi}{7}\right)}}+\frac{1}{\sqrt[3]{\tan\left(\frac{8\pi}{7}\right)}}=$$

$$\left(-\frac{1}{\sqrt[18]{7}}\right)\left(\sqrt[3]{-\sqrt[3]{49}+6+3\left(\sqrt[3]{3\sqrt[3]{7}-3\sqrt[3]{49}+5}+\sqrt[3]{3\sqrt[3]{7}-3\sqrt[3]{49}-3}\right)}\right). \quad (10)$$

The Equations (1)–(10) can be derived using the following result of Ramanujan [1–3].

Theorem 1.5. *Let $\alpha, \beta,$ and γ denote the roots of a cubic equation*

$$x^3-ax^2+bx-1=0. \quad (11)$$

If α, β, γ are real and the values of the cubic roots of these numbers below are real, then

$$\sqrt[3]{\alpha}+\sqrt[3]{\beta}+\sqrt[3]{\gamma}=\sqrt[3]{a+6+3t}, \quad (12)$$

and

$$\frac{1}{\sqrt[3]{\alpha}}+\frac{1}{\sqrt[3]{\beta}}+\frac{1}{\sqrt[3]{\gamma}}=\sqrt[3]{b+6+3t}, \quad (13)$$

where t is a real root of the equation

$$t^3-3(a+b+3)t-(ab+6(a+b)+9)=0. \quad (14)$$

We assume henceforward that all the discussed cubic equations of the form (11) satisfy the assumptions given in the theorem above. We will call Equation (14) an associated Ramanujan Equation. The Ramanujan equation can be solved using Cardano formula. For our purpose, we reformulate it in the following form.

Theorem 1.6 ([4]). *For a cubic equation,*

$$t^3 + pt + q = 0. \tag{15}$$

a real solution is given by

$$t = \left(\frac{1}{\sqrt[3]{2}}\right)\sqrt[3]{-q + \sqrt{q^2 + \frac{4p^3}{27}}} + \left(\frac{1}{\sqrt[3]{2}}\right)\sqrt[3]{-q - \sqrt{q^2 + \frac{4p^3}{27}}}. \tag{16}$$

When we studied the term inside the square root in the Cardano formula (Equation (16)) we found out that this term is a complete square and it can be expressed in term of roots of the Equation (11). We believe that this formula is also new. This formula makes the computation of the solution for Ramanujan equation much easier when the coefficients of a cubic equation are not integers.

Theorem 1.7. *Let $\alpha, \beta,$ and γ denote the roots of a cubic equation*

$$x^3 - ax^2 + bx - 1 = 0, \tag{17}$$

where a, b are possibly complex numbers and let

$$x^3 + px + q = 0 \tag{18}$$

be the associated Ramanujan equation. Then

$$q^2 + \frac{4p^3}{27} = \left(\left(\frac{\alpha}{\beta} + \frac{\beta}{\gamma} + \frac{\gamma}{\alpha}\right) - \left(\frac{\beta}{\alpha} + \frac{\gamma}{\beta} + \frac{\alpha}{\gamma}\right)\right)^2. \tag{19}$$

Our methods and proofs are completely different from these in [6]. In the next Section 2, we will introduce some notations and prove some basic properties of trigonometric functions. The proof for Theorem 1.7 is given in the Section 3. We then prove the formula for sums of cubic roots for sine and tangent functions in Sections 4, 5, 6, 7, respectively.

2 Notations and properties of sine and tangent functions

For the rest of this paper, for convenience, let $\theta = \frac{\pi}{9}$ and let $\delta = \frac{\pi}{7}$.

We will use following well-known results (see [5]).

Proposition 2.1. *The numbers $\{\sin(\theta), \sin(2\theta), \sin(14\theta)\}$ are the roots of the equation*

$$x^3 - \frac{3}{4}x + \frac{1}{8}\sqrt{3} = 0 \tag{20}$$

The numbers $\{\cos 2\theta, \cos 4\theta, \cos 8\theta\}$ are the roots of the equation

$$x^3 - \frac{3}{4}x + \frac{1}{8} = 0. \tag{21}$$

The numbers $\{\tan(\theta), \tan(4\theta), \tan(7\theta)\}$ are the roots of the equation

$$x^3 - 3\sqrt{3}x^2 - 3x + \sqrt{3} = 0. \tag{22}$$

Note that we cannot use

$$\{\sin(2\theta), \sin(4\theta), \sin(8\theta)\}$$

for Equation (20) nor

$$\{\tan(2\theta), \tan(4\theta), \tan(8\theta)\}$$

for Equation (22). Note also for sine function, we rotate

$$\{\sin(\theta), \sin(2\theta), \sin(14\theta)\}$$

and for tangent function, we rotate

$$\{\tan(\theta), \tan(4\theta), \tan(7\theta)\}.$$

We will use following well-known results (see [5]).

Proposition 2.2. *The numbers* $\{\sin(2\delta), \sin(4\delta), \sin(8\delta)\}$ *are the roots of the equation*

$$x^3 - \frac{\sqrt{7}}{2}x^2 + \frac{\sqrt{7}}{8} = 0.$$

The numbers $\{\cos 2\delta, \cos 4\delta, \cos 8\delta\}$ *are the roots of the equation*

$$x^3 + \frac{1}{2}x^2 - \frac{1}{2}x - \frac{1}{8} = 0.$$

The numbers $\{\tan 2\delta, \tan 4\delta, \tan 8\delta\}$ *are the roots of the equation*

$$x^3 + \sqrt{7}x^2 - 7x + \sqrt{7} = 0.$$

Definition 2.3. *Let* $\{\alpha, \beta, \gamma\}$ *be three numbers. For a function* $f(x)$ *with one variable, let*

$$\sum f(\alpha) = f(\alpha) + f(\beta) + f(\gamma).$$

For a function $f(x, y)$ *with two variables, let*

$$\sum f(\alpha, \beta) = f(\alpha, \beta) + f(\beta, \gamma) + f(\gamma, \alpha).$$

For a function $f(x, y, z)$ *with three variables, let*

$$\sum f(\alpha, \beta, \gamma) = f(\alpha, \beta, \gamma) + f(\beta, \gamma, \alpha) + f(\gamma, \alpha, \beta).$$

Definition 2.4. *For an integer* m, *let*

$$S_9(m) = \sum \sin^m(\theta),$$
$$C_9(m) = \sum \cos^m(2\theta),$$
$$T_9(m) = \sum \tan^m(\theta).$$

Definition 2.5. *For integers m, n, let*

$$U_9(m,n) = \sum \sin^m(\theta)\sin^n(2\theta)$$
$$V_9(m,n) = \sum \cos^m(2\theta)\cos^n(4\theta),$$
$$W_9(m,n) = \sum \tan^m(\theta)\tan^n(4\theta).$$

Definition 2.6. *Let*

$$P_9 = \sin(\theta)\sin(2\theta)\sin(14\theta) = -\frac{\sqrt{3}}{8},$$
$$Q_9 = \cos(2\theta)\cos(4\theta)\cos(8\theta) = -\frac{1}{8},$$
$$R_9 = \tan(\theta)\tan(4\theta)\tan(7\theta) = -\sqrt{3}.$$

Proposition 2.7. *With the above notations, for $n = 0, \ldots, 3$,*

$$T_9(n) = 3, 3\sqrt{3}, 33, 105\sqrt{3}, \ldots$$
$$T_9(-n) = 3, \sqrt{3}, 9, 11\sqrt{3}, \ldots$$

Proof. Note that $T_9(n)$ satisfy the following recurrence relations:

$$T_9(n) = 3\sqrt{3}T_9(n-1) + 3T_9(n-2) - \sqrt{3}T_9(n-3),$$
$$T_9(-n) = \sqrt{3}T_9(-n+1) + 3T_9(-n+2) - \frac{\sqrt{3}}{3}T_9(-n+3).$$

$$T_9(0) = 3.$$
$$T_9(1) = 3\sqrt{3}.$$
$$\begin{aligned} T_9(2) &= T_9(1)^2 - 2W_9(1,1) \\ &= (3\sqrt{3})^2 - 2(-3) \\ &= 33. \end{aligned}$$
$$\begin{aligned} T_9(3) &= 3\sqrt{3}T_9(2) + 3T_9(1) - \sqrt{3}T_9(0) \\ &= 3\sqrt{3}(33) + 3(3\sqrt{3}) - \sqrt{3}(3) \\ &= 105\sqrt{3}. \end{aligned}$$
$$\begin{aligned} T_9(-1) &= \sqrt{3}T_9(0) + 3T_9(1) - \frac{\sqrt{3}}{3}T_9(2) \\ &= \sqrt{3}(3) + 3(3\sqrt{3}) - \frac{\sqrt{3}}{3}(33) \\ &= \sqrt{3}. \end{aligned}$$
$$\begin{aligned} T_9(-2) &= \sqrt{3}T_9(-1) + 3T_9(0) - \frac{\sqrt{3}}{3}T_9(1) \\ &= \sqrt{3}(\sqrt{3}) + 3(3) - \frac{\sqrt{3}}{3}(3\sqrt{3}) \\ &= 9. \end{aligned}$$

$$\begin{aligned}
T_9(-3) &= \sqrt{3}T_9(-2) + 3T_9(-1) - \frac{\sqrt{3}}{3}T_9(0)\\
&= \sqrt{3}(9) + 3(\sqrt{3}) - (3)\frac{\sqrt{3}}{3}\\
&= 11\sqrt{3}.
\end{aligned}$$
□

Proposition 2.8.

$$\begin{aligned}
W_9(1,2) &= 9\sqrt{3},\\
W_9(2,1) &= -15\sqrt{3}.
\end{aligned}$$

Proof.

$$\begin{aligned}
W_9(1,2) + W_9(2,1) &= \sum \tan(\theta)\tan^2(4\theta) + \sum \tan^2(\theta)\tan(4\theta)\\
&= \left(\sum \tan(\theta)\right)\left(\sum \tan(\theta)\tan(4\theta)\right) - 3\tan(\theta)\tan(4\theta)\tan(7\theta)\\
&= (3\sqrt{3})(-3) - 3(-\sqrt{3})\\
&= -6\sqrt{3}.
\end{aligned}$$

$$\begin{aligned}
W_9(1,2)W_9(2,1) &= \left(\sum \tan(\theta)\tan^2(4\theta)\right)\left(\sum \tan^2(\theta)\tan(4\theta)\right)\\
&= \left(R_9^2\right)\left(\sum \frac{\tan(4\theta)}{\tan(7\theta)}\right)\left(\sum \frac{\tan(\theta)}{\tan(7\theta)}\right)\\
&= 3\left(3 + \frac{1}{R_9}\sum \tan^3(\theta) + R_9 \sum \frac{1}{\tan^3(\theta)}\right)\\
&= 3\left(3 + \frac{1}{R_9}T(3) + R_9T_9(-3)\right)\\
&= 3\left(3 + \frac{-1}{\sqrt{3}}(105\sqrt{3}) + (-\sqrt{3})(11\sqrt{3})\right)\\
&= 3(3 - 105 - 33)\\
&= -405
\end{aligned}$$

Then $\{W_9(1,2), W_9(2,1)\}$ are the roots of quadratic equation

$$x^2 - 6\sqrt{3}x - 405 = 0. \tag{23}$$

The roots for Equation (23) are $\{9\sqrt{3}, -15\sqrt{3}\}$.

With a help from calculator, we have

$$W_9(1,2) = 9\sqrt{3}, \quad W_9(2,1) = -15\sqrt{3}.$$
□

Similarly, for $\pi/7$ we have

Definition 2.9. *For an integer m, let*

$$\begin{aligned}
S_7(m) &= \sum \sin^m(2\delta),\\
C_7(m) &= \sum \cos^m(2\delta),\\
T_7(m) &= \sum \tan^m(2\delta).
\end{aligned}$$

Definition 2.10. *For integers m, n, let*

$$\begin{aligned}
U_7(m,n) &= \sum \sin^m(2\delta)\sin^n(4\delta),\\
V_7(m,n) &= \sum \cos^m(2\delta)\cos^n(4\delta),\\
W_7(m,n) &= \sum \tan^m(2\delta)\tan^n(4\delta).
\end{aligned}$$

Definition 2.11. *Let*

$$\begin{aligned}
P_7 &= \sin(2\delta)\sin(4\delta)\sin(8\delta) = -\frac{\sqrt{7}}{8},\\
Q_7 &= \cos(2\delta)\cos(4\delta)\cos(8\delta) = \frac{1}{8},\\
R_7 &= \tan(2\delta)\tan(4\delta)\tan(8\delta) = -\sqrt{7}.
\end{aligned}$$

Proposition 2.12. *With the above notations, for $n = 0, \ldots, 3$,*

$$\begin{aligned}
T_7(n) &= 3, -\sqrt{7}, 21, -31\sqrt{7}, \ldots\\
T_7(-n) &= 3, \sqrt{7}, 5, \frac{25\sqrt{7}}{7}, \ldots
\end{aligned}$$

Proof. Note that $T_7(n)$ satisfy the following recurrence relations:

$$\begin{aligned}
T_7(n) &= -\sqrt{7}T_7(n-1) + 7_7T(n-2) - \sqrt{7}T_7(n-3),\\
T_7(-n) &= \sqrt{7}T_7(-n+1) - T_7(-n+2) - \frac{\sqrt{7}}{7}T_7(-n+3).
\end{aligned}$$

$$\begin{aligned}
T_7(2) &= T_7(1)^2 - 2W_7(1,1)\\
&= (-\sqrt{7})^2 - 2(-7)\\
&= 21.
\end{aligned}$$

$$\begin{aligned}
T_7(3) &= -\sqrt{7}T_7(2) + 7T_7(1) - \sqrt{7}T_7(0)\\
&= (-21 - 7 - 3)\sqrt{7}\\
&= -31\sqrt{7}.
\end{aligned}$$

$$\begin{aligned}
T_7(-1) &= \sqrt{7}T_7(0) - T_7(1) - \frac{\sqrt{7}}{7}T_7(2)\\
&= 3\sqrt{7} + \sqrt{7} - 21\left(\frac{\sqrt{7}}{7}\right)\\
&= \sqrt{7}.
\end{aligned}$$

$$\begin{aligned}
T_7(-2) &= \sqrt{7}T_7(-1) - T_7(0) - \frac{\sqrt{7}}{7}T_7(1)\\
&= \sqrt{7}\sqrt{7} - 3 - \frac{\sqrt{7}}{7}(-\sqrt{7})\\
&= 7 - 3 + 1\\
&= 5.
\end{aligned}$$

$$
\begin{aligned}
T_7(-3) &= \sqrt{7}T_7(-2) - T_7(-1) - \frac{\sqrt{7}}{7}T_7(0) \\
&= 5\sqrt{7} - \sqrt{7} - 3\frac{\sqrt{7}}{7} \\
&= (35 - 7 - 3)\frac{\sqrt{7}}{7} \\
&= \frac{25\sqrt{7}}{7}. \qquad \square
\end{aligned}
$$

Proposition 2.13.

$$
\begin{aligned}
W_7(1,2) &= 9\sqrt{7}, \\
W_7(2,1) &= \sqrt{7}.
\end{aligned}
$$

Proof.

$$
\begin{aligned}
W_7(1,2) + W_7(2,1) &= T_7(1)W_7(1,1) - 3R_7 \\
&= (-\sqrt{7})(-7) - 3(-\sqrt{7}) \\
&= 7\sqrt{7} + 3\sqrt{7} \\
&= 10\sqrt{7}.
\end{aligned}
$$

$$
\begin{aligned}
W_7(1,2)W_7(2,1) &= \left(R_7^2\right)\left(\sum \frac{\tan(2\delta)}{\tan(4\delta)}\right)\left(\sum \frac{\tan(4\delta)}{\tan(2\delta)}\right) \\
&= 7\left(3 + \frac{1}{R_7}\sum \tan^3(2\delta) + R_7 \sum \frac{1}{\tan^3(2\delta)}\right) \\
&= 7\left(3 + \frac{1}{R_7}T_7(3) + R_7T_7(-3)\right) \\
&= 7\left(3 - \frac{1}{\sqrt{7}}T_7(3) + (-\sqrt{7})T_7(-3)\right) \\
&= 7\left(3 - \frac{1}{\sqrt{7}}(-31\sqrt{7}) + (-\sqrt{7})(\frac{25\sqrt{7}}{7})\right) \\
&= 7(3 + 31 - 25) \\
&= 63.
\end{aligned}
$$

Then $\{W_7(1,2), W_7(2,1)\}$ are the roots of quadratic equation

$$
x^2 - 10\sqrt{7}x + 63 = 0. \tag{24}
$$

The roots for Equation (24) are $\{9\sqrt{7}, \sqrt{7}\}$.

With a help from calculator, we have

$$
W_7(1,2) = 9\sqrt{7}, W_7(2,1) = \sqrt{7}. \qquad \square
$$

3 Proof of Theorem 1.7

Proof. Let

$$u = \sum \frac{\alpha}{\beta}, \quad v = \sum \frac{\beta}{\alpha}.$$

Then

$$\begin{aligned} u + v &= \sum \frac{\alpha}{\beta} + \sum \frac{\beta}{\alpha} \\ &= \left(\sum \alpha\right)\left(\sum \frac{1}{\alpha}\right) - 3 \\ &= ab - 3. \end{aligned}$$

$$\begin{aligned} \sum \alpha^3 &= (\sum \alpha)^3 - 3\sum \alpha^2\beta - 3\sum \alpha\beta^2 - 6\alpha\beta\gamma \\ &= (\sum \alpha)^3 - 3\sum \frac{\alpha}{\beta} - 3\sum \frac{\beta}{\alpha} - 6 \\ &= a^3 - 3(ab-3) - 6 \\ &= a^3 - 3ab + 3. \end{aligned}$$

$$\begin{aligned} \sum \frac{1}{\alpha^3} &= (\sum \frac{1}{\alpha})^3 - 3\sum \frac{1}{\alpha^2\beta} - 3\sum \frac{1}{\alpha\beta^2} - \frac{6}{\alpha\beta\gamma} \\ &= (\sum \frac{1}{\alpha})^3 - 3\sum \frac{\beta}{\alpha} - 3\sum \frac{\alpha}{\beta} - 6 \\ &= b^3 - 3(ab-3) - 6 \\ &= b^3 - 3ab + 3. \end{aligned}$$

$$\begin{aligned} uv &= \left(\sum \frac{\alpha}{\beta}\right)\left(\sum \frac{\beta}{\alpha}\right) \\ &= \left(\frac{\alpha}{\beta}\right)\left(\sum \frac{\beta}{\alpha}\right) + \left(\frac{\beta}{\gamma}\right)\left(\sum \frac{\beta}{\alpha}\right) + \left(\frac{\gamma}{\alpha}\right)\left(\sum \frac{\beta}{\alpha}\right) \\ &= 3 + \sum \frac{\alpha^2}{\beta\gamma} + \sum \frac{\beta\gamma}{\alpha^2} \\ &= 3 + \sum \alpha^3 + \sum \frac{1}{\alpha^3} \\ &= 3 + (a^3 - 3ab + 3) + (b^3 - 3ab + 3) \\ &= a^3 + b^3 - 6ab + 9. \end{aligned}$$

Now we obtain

$$\begin{aligned} (u-v)^2 &= (u+v)^2 - 4uv \\ &= (ab-3)^2 - 4(a^3 + b^3 - 6ab + 9) \\ &= ((ab)^2 - 6ab + 9) - 4(a^3 + b^3 - 6ab + 9) \\ &= (ab)^2 - 4a^3 - 4b^3 + 18ab - 27, \end{aligned}$$

and, on the other hand,

$$\begin{aligned}
q^2 + \frac{4p^3}{27} &= (ab + 6(a+b) + 9)^2 - 4(a+b+3)^3 \\
&= (ab)^2 + 36(a+b)^2 + 81 + 12ab(a+b) + 108(a+b) + 18ab \\
&\quad - 4(a^3 + b^3 + 27 + 3a^2b + b^2a + 9a^2 + 9b^2 + 2a + 27b + 18ab) \\
&= (ab)^2 + 36 * (a^2 + 2ab + b^2) + 81 + 12ab(a+b) + 108(a+b) + 18ab \\
&\quad - (4a^3 + 4b^3 + 108 + 12a^2b + 12b^2a + 36a^2 + 36b^2 + 108a + 108b + 72ab) \\
&= (ab)^2 - 4a^3 - 4b^3 + 18ab - 27 \\
&= (u-v)^2.
\end{aligned}$$

This completes the proof. □

4 Proof of Theorem 1.1

In this section, we will prove Theorem 1.1.

Proof. Let

$$\alpha = -\frac{2}{\sqrt[6]{3}}\sin(\theta), \beta = -\frac{2}{\sqrt[6]{3}}\sin(2\theta), \gamma = -\frac{2}{\sqrt[6]{3}}\sin(14\theta).$$

$$\begin{aligned}
\sum \alpha &= \left(-\frac{2}{\sqrt[6]{3}}\right)(\sin(\theta) + \sin(2\theta) + \sin(14\theta)) \\
&= \left(-\frac{2}{\sqrt[6]{3}}\right)(0) \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\sum \alpha\beta &= \left(\frac{4}{\sqrt[6]{9}}\right)(\sin(\theta)\sin(2\theta) + \sin(2\theta)\sin(14\theta) + \sin(14\theta))\sin(\theta)) \\
&= \left(\frac{4}{\sqrt[6]{9}}\right)\left(-\frac{3}{4}\right) \\
&= -\frac{3}{\sqrt[3]{3}} \\
&= -\sqrt[3]{9}.
\end{aligned}$$

$$\begin{aligned}
\alpha\beta\gamma &= \left(-\frac{8}{\sqrt{3}}\right)(\sin(\theta)\sin(2\theta)\sin(14\theta)) \\
&= \left(-\frac{8}{\sqrt{3}}\right)\left(-\frac{\sqrt{3}}{8}\right) \\
&= 1.
\end{aligned}$$

It follows that $\{\alpha, \beta, \gamma\}$ are roots of the equation:

$$x^3 - \sqrt[3]{9}x - 1 = 0. \tag{25}$$

The associated Ramanujan equation is equal to

$$t^3 + pt + q = 0, \tag{26}$$

where

$$\begin{aligned} p &= 3\sqrt[3]{9} - 9, \\ q &= 6\sqrt[3]{9} - 9. \end{aligned}$$

We now compute $\sum \frac{\alpha}{\beta}$ and $\sum \frac{\beta}{\alpha}$.

$$\begin{aligned} u &= \sum \frac{\alpha}{\beta} \\ &= \frac{\sin(\theta)}{\sin(2\theta)} + \frac{\sin(2\theta)}{\sin(14\theta)} + \frac{\sin(14\theta)}{\sin(\theta)} \\ &= \frac{\sin(\theta)}{\sin(2\theta)} + \frac{\sin(7\theta)}{\sin(14\theta)} - \frac{\sin(4\theta)}{\sin(8\theta)} \\ &= \frac{\sin(\theta)}{2\sin(\theta)\cos(\theta)} + \frac{\sin(7\theta)}{2\sin(7\theta)\cos(7\theta)} - \frac{\sin(4\theta)}{2\sin(4\theta)\cos(4\theta)} \\ &= \frac{1}{2}\left(\frac{1}{\cos(\theta)} + \frac{1}{\cos(7\theta)} - \frac{1}{\cos(4\theta)}\right) \\ &= -\frac{1}{2}\sum \frac{1}{\cos(2\theta)} \\ &= -\frac{1}{2}\sum \frac{\cos(2\theta)\cos(4\theta)}{\cos(2\theta)\cos(4\theta)\cos(8\theta)} \\ &= \left(-\frac{1}{2}\right)\left(\frac{1}{Q_9}\right)\sum \cos(4\theta)\cos(8\theta) \\ &= \left(-\frac{1}{2}\right)(-8)\left(-\frac{3}{4}\right) \\ &= -3. \end{aligned}$$

In the following proofs, we use double angle formula for sine function.

$$\begin{aligned} v &= \sum \frac{\beta}{\alpha} \\ &= \frac{\sin(2\theta)}{\sin(\theta)} + \frac{\sin(14\theta)}{\sin(2\theta)} + \frac{\sin(\theta)}{\sin(14\theta)} \\ &= \frac{\sin(2\theta)}{\sin(\theta)} + \frac{\sin(14\theta)}{\sin(7\theta)} - \frac{\sin(8\theta)}{\sin(4\theta)} \\ &= \frac{2\sin(\theta)\cos(\theta)}{\sin(\theta)} + \frac{2\sin(7\theta)\cos(7\theta)}{\sin(7\theta)} - \frac{2\sin(4\theta)\cos(4\theta)}{\sin(4\theta)} \\ &= 2\left(\cos(\theta) + \cos(7\theta) - \cos(4\theta)\right) \\ &= -2\sum \cos(2\theta) \\ &= 0. \end{aligned}$$

By Theorem 1.7,

$$q^2 + \frac{4p^3}{27} = \left(\sum \frac{\alpha}{\beta} - \sum \frac{\beta}{\alpha}\right)^2 = 9.$$

By Theorem 1.6, a real solution for the associated Ramanujan equation is given by

$$\begin{aligned}
t &= \frac{1}{\sqrt[3]{2}}\sqrt[3]{-q+\sqrt{q^2+\frac{4p^3}{27}}} + \frac{1}{\sqrt[3]{2}}\sqrt[3]{-q-\sqrt{q^2+\frac{4p^3}{27}}} \\
&= \frac{1}{\sqrt[3]{2}}\sqrt[3]{-(6\sqrt[3]{9}-9)+3} + \frac{1}{\sqrt[3]{2}}\sqrt[3]{-(6\sqrt[3]{9}-9)-3} \\
&= \frac{1}{\sqrt[3]{2}}\sqrt[3]{-6\sqrt[3]{9}+9+3} + \frac{1}{\sqrt[3]{2}}\sqrt[3]{-6\sqrt[3]{9}+9-3} \\
&= \frac{1}{\sqrt[3]{2}}\sqrt[3]{-6\sqrt[3]{9}+12} + \frac{1}{\sqrt[3]{2}}\sqrt[3]{-6\sqrt[3]{9}+6} \\
&= \frac{1}{\sqrt[3]{2}}\sqrt[3]{12-6\sqrt[3]{9}} + \frac{1}{\sqrt[3]{2}}\sqrt[3]{6-6\sqrt[3]{9}} \\
&= \sqrt[3]{6-3\sqrt[3]{9}} + \sqrt[3]{3-3\sqrt[3]{9}}.
\end{aligned}$$

By Theorem 1.5,

$$\sum \sqrt[3]{\alpha} = \sqrt[3]{6+3\left(\sqrt[3]{6-3\sqrt[3]{9}} + \sqrt[3]{3-3\sqrt[3]{9}}\right)}.$$

and

$$\begin{aligned}
\sum \frac{1}{\sqrt[3]{\alpha}} &= \sqrt[3]{b+6+3t} \\
&= \sqrt[3]{-\sqrt[3]{9}+6+3\left(\sqrt[3]{6-3\sqrt[3]{9}} + \sqrt[3]{3-3\sqrt[3]{9}}\right)}
\end{aligned}$$

This implies Equations (3) and (4) in Theorem 1.1. □

5 Proof of Theorem 1.2

In this section, we will prove Theorem 1.2.

Proof. Let

$$\alpha = -\frac{1}{\sqrt[6]{3}}\tan(\theta), \beta = -\frac{1}{\sqrt[6]{3}}\tan(4\theta), \gamma = -\frac{1}{\sqrt[6]{3}}\tan(7\theta).$$

$$\begin{aligned}
a &= \sum \alpha \\
&= \left(-\frac{1}{\sqrt[6]{3}}\right)(\tan(\theta)+\tan(4\theta)+\tan(7\theta))
\end{aligned}$$

$$
\begin{aligned}
&= \left(-\frac{1}{\sqrt[6]{3}}\right)\left(3\sqrt{3}\right) \\
&= \left(-\frac{3\sqrt[6]{27}}{\sqrt[6]{3}}\right) \\
&= -3\sqrt[6]{9} \\
&= -3\sqrt[3]{3}.
\end{aligned}
$$

$$
\begin{aligned}
b &= \sum \alpha\beta \\
&= \left(\frac{1}{\sqrt[3]{3}}\right)\sum \tan(\theta)\tan(4\theta) \\
&= \left(\frac{1}{\sqrt[3]{3}}\right)(-3) \\
&= -\sqrt[3]{9}.
\end{aligned}
$$

$$
\begin{aligned}
\alpha\beta\gamma &= \left(-\frac{1}{\sqrt{3}}\right)(\tan(\theta)\tan(4\theta)\tan(7\theta)) \\
&= 1.
\end{aligned}
$$

It follows that $\{\alpha, \beta, \gamma\}$ are roots of the equation:

$$x^3 + 3\sqrt[3]{3}x^2 - \sqrt[3]{9}x - 1 = 0. \tag{27}$$

The associated Ramanujan equation is equal to

$$t^3 + pt + q = 0, \tag{28}$$

where

$$
\begin{aligned}
&p = 3(3\sqrt[3]{3} + \sqrt[3]{9} - 3), \\
&q = -(18 - 6(3\sqrt[3]{3} - \sqrt[3]{9})).
\end{aligned}
$$

We now compute $\sum \frac{\alpha}{\beta}$ and $\sum \frac{\beta}{\alpha}$

$$
\begin{aligned}
u &= \sum \frac{\alpha}{\beta} \\
&= \sum \frac{\tan(\theta)}{\tan(4\theta)} \\
&= \sum \frac{\tan^2(\theta)\tan(7\theta)}{\tan(\theta)\tan(4\theta)\tan(7\theta)} \\
&= \frac{1}{R_9}\sum \tan^2(\theta)\tan(7\theta) \\
&= -\left(\frac{1}{\sqrt{3}}\right)W_9(1,2) \\
&= -\left(\frac{1}{\sqrt{3}}\right)(9\sqrt{3}) \\
&= -9.
\end{aligned}
$$

$$
\begin{aligned}
v &= \sum \frac{\beta}{\alpha} \\
&= \sum \frac{\tan(4\theta)}{\tan(\theta)} \\
&= \sum \frac{\tan^2(4\theta)\tan(7\theta)}{\tan(\theta)\tan(4\theta)\tan(7\theta)} \\
&= \frac{1}{R_9}\sum \tan^2(4\theta)\tan(7\theta) \\
&= -\left(\frac{1}{\sqrt{3}}\right)\sum \tan^2(4\theta)\tan(7\theta) \\
&= -\left(\frac{1}{\sqrt{3}}\right) W_9(2,1) \\
&= 15.
\end{aligned}
$$

By Theorem 1.7,

$$
q^2 + \frac{4p^3}{27} = \left(\sum \frac{\alpha}{\beta} - \sum \frac{\beta}{\alpha}\right)^2 = 24^2.
$$

$$
\begin{aligned}
t &= \frac{1}{\sqrt[3]{2}}\sqrt[3]{-q+\sqrt{q^2+\frac{4p^3}{27}}} + \frac{1}{\sqrt[3]{2}}\sqrt[3]{-q-\sqrt{q^2+\frac{4p^3}{27}}} \\
&= \frac{1}{\sqrt[3]{2}}\sqrt[3]{-q+24} + \frac{1}{\sqrt[3]{2}}\sqrt[3]{-q-24} \\
&= \frac{1}{\sqrt[3]{2}}\sqrt[3]{42-6(3\sqrt[3]{3}-\sqrt[3]{9})} + \frac{1}{\sqrt[3]{2}}\sqrt[3]{-6-6(3\sqrt[3]{3}-\sqrt[3]{9})} \\
&= \sqrt[3]{21-3(3\sqrt[3]{3}-\sqrt[3]{9})} + \sqrt[3]{-3-3(3\sqrt[3]{3}-\sqrt[3]{9})} \\
&= \sqrt[3]{21-3(3\sqrt[3]{3}-\sqrt[3]{9})} - \sqrt[3]{3+3(3\sqrt[3]{3}+\sqrt[3]{9})}
\end{aligned}
$$

By Theorem 1.5,

$$
\begin{aligned}
\sum \sqrt[3]{\alpha} &= \sqrt[3]{a+6+3t} \\
&= \sqrt[3]{(-3\sqrt[3]{3})+6+3(\sqrt[3]{21-3(3\sqrt[3]{3}-\sqrt[3]{9})}-\sqrt[3]{3+3(3\sqrt[3]{3}+\sqrt[3]{9})})} \\
&= \sqrt[3]{-3\sqrt[3]{3}+6+3(\sqrt[3]{21-3(3\sqrt[3]{3}-\sqrt[3]{9})}-\sqrt[3]{3+3(3\sqrt[3]{3}+\sqrt[3]{9})})}
\end{aligned}
$$

and

$$
\begin{aligned}
\sum \frac{1}{\sqrt[3]{\alpha}} &= \sqrt[3]{b+6+3t} \\
&= \sqrt[3]{(-\sqrt[3]{9})+6+3(\sqrt[3]{21-3(3\sqrt[3]{3}-\sqrt[3]{9})}-\sqrt[3]{3+3(3\sqrt[3]{3}+\sqrt[3]{9})})} \\
&= \sqrt[3]{-\sqrt[3]{9})+6+3(\sqrt[3]{21-3(3\sqrt[3]{3}-\sqrt[3]{9})}-\sqrt[3]{3+3(3\sqrt[3]{3}+\sqrt[3]{9})})}.
\end{aligned}
$$

This implies Equations (5) and (6) in Theorem 1.2. □

6 Proof of Theorem 1.3

To further illustrate out method, in this section, we will prove Theorem 1.3, which is similar to the proof of Theorem 1.1.

Proof. Let
$$\alpha = -\frac{2}{\sqrt[6]{7}}\sin(2\delta), \beta = -\frac{2}{\sqrt[6]{7}}\sin(4\delta), \gamma = -\frac{2}{\sqrt[6]{7}}\sin(8\delta).$$

$$\begin{aligned}\sum \alpha &= \left(-\frac{2}{\sqrt[6]{7}}\right)(\sin(2\delta)+\sin(4\delta)+\sin(8\delta))\\ &= \left(-\frac{2}{\sqrt[6]{7}}\right)\left(\frac{\sqrt{7}}{2}\right)\\ &= -\sqrt[3]{7}.\end{aligned}$$

$$\begin{aligned}\sum \alpha\beta &= \left(\frac{4}{\sqrt[3]{7}}\right)(\sin(2\delta)\sin(4\delta)+\sin(4\delta)\sin(4\delta)+\sin(8\delta))\sin(4\delta))\\ &= 0.\end{aligned}$$

$$\begin{aligned}\alpha\beta\gamma &= \left(-\frac{8}{\sqrt{7}}\right)(\sin(2\delta)\sin(4\delta)\sin(8\delta))\\ &= 1.\end{aligned}$$

It follows that $\{\alpha, \beta, \gamma\}$ are roots of the equation:
$$x^3 + \sqrt[3]{7}x^2 - 1 = 0. \tag{29}$$

The associated Ramanujan equation is equal to
$$t^3 + pt + q = 0, \tag{30}$$

where
$$\begin{aligned}p &= 3(\sqrt[3]{7} - 3),\\ q &= 6\sqrt[3]{7} - 9.\end{aligned}$$

We now compute $\sum \frac{\alpha}{\beta}$ and $\sum \frac{\beta}{\alpha}$.

$$\begin{aligned}u &= \sum \frac{\alpha}{\beta}\\ &= \sum \frac{\sin(2\delta)}{\sin(4\delta)}\\ &= \sum \frac{\sin(2\delta)}{2\sin(2\delta)\cos(2\delta)}\\ &= \frac{1}{2}\sum \frac{\cos(2\delta)\cos(4\delta)}{\cos(2\delta)\cos(4\delta)\cos(8\delta)}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2Q_7}\sum \cos(2\delta)\cos(4\delta) \\
&= 4\sum \cos(2\delta)\cos(4\delta) \\
&= -2.
\end{aligned}
$$

$$
\begin{aligned}
v &= \sum \frac{\beta}{\alpha} \\
&= \sum \frac{\sin(4\delta)}{\sin(2\delta)} \\
&= \sum \frac{2\sin(2\delta)\cos(2\delta)}{\sin(2\delta)} \\
&= 2\sum \cos(2\delta) \\
&= -1.
\end{aligned}
$$

By Theorem 1.7,

$$
q^2 + \frac{4p^3}{27} = \left(\sum \frac{\alpha}{\beta} - \sum \frac{\beta}{\alpha}\right)^2 = 1.
$$

By Theorem 1.6, a real solution for the associated Ramanujan equation is given by

$$
\begin{aligned}
t &= \frac{1}{\sqrt[3]{2}}\sqrt[3]{-q+\sqrt{q^2+\frac{4p^3}{27}}} + \frac{1}{\sqrt[3]{2}}\sqrt[3]{-q-\sqrt{q^2+\frac{4p^3}{27}}} \\
&= \frac{1}{\sqrt[3]{2}}\sqrt[3]{-q+\sqrt{1}} + \frac{1}{\sqrt[3]{2}}\sqrt[3]{-q-\sqrt{1}} \\
&= \frac{1}{\sqrt[3]{2}}\sqrt[3]{10-6\sqrt[3]{7}} + \frac{1}{\sqrt[3]{2}}\sqrt[3]{8-6\sqrt[3]{7}} \\
&= \sqrt[3]{5-3\sqrt[3]{7}} + \sqrt[3]{4-3\sqrt[3]{7}}.
\end{aligned}
$$

By Theorem 1.5,

$$
\begin{aligned}
\sum \sqrt[3]{\alpha} &= \sqrt[3]{a+6+3t} \\
&= \sqrt[3]{-\sqrt[3]{7}+6+3\left(\sqrt[3]{5-3\sqrt[3]{7}}+\sqrt[3]{4-3\sqrt[3]{7}}\right)} \\
&= \sqrt[3]{-\sqrt[3]{7}+6+3\sqrt[3]{5-3\sqrt[3]{7}}+3\sqrt[3]{4-3\sqrt[3]{7}}}
\end{aligned}
$$

and

$$
\begin{aligned}
\sum \frac{1}{\sqrt[3]{\alpha}} &= \sqrt[3]{b+6+3t} \\
&= \sqrt[3]{6+3\sqrt[3]{5-3\sqrt[3]{7}}+3\sqrt[3]{4-3\sqrt[3]{7}}}
\end{aligned}
$$

This implies Equations (7) and (8) in Theorem 1.3. $\square$

7 Proof of Theorem 1.4

To further illustrate out method, in this section, we will prove Theorem 1.4, which is similar to the proof of Theorem 1.2.

Proof.

$$\begin{aligned}
\sum \alpha &= \left(-\frac{1}{\sqrt[6]{7}}\right)(\tan(2\delta)+\tan(4\delta)+\tan(8\delta)) \\
&= \left(-\frac{1}{\sqrt[6]{7}}\right)\left(-\sqrt{7}\right) \\
&= \sqrt[3]{7}. \\
\sum \alpha\beta &= \left(\frac{1}{\sqrt[3]{7}}\right)(\tan(2\delta)\tan(4\delta)+\tan(4\delta)\tan(4\delta)+\tan(8\delta))\tan(4\delta)) \\
&= \left(\frac{1}{\sqrt[3]{7}}\right)(-7) \\
&= -\sqrt[3]{49}. \\
\alpha\beta\gamma &= \left(-\frac{1}{\sqrt{7}}\right)(\tan(2\delta)\tan(4\delta)\tan(8\delta)) \\
&= 1.
\end{aligned}$$

It follows that $\{\alpha, \beta, \gamma\}$ are roots of the equation:

$$x^3 - \sqrt[3]{7}x^2 - \sqrt[3]{49}x - 1 = 0. \tag{31}$$

The associated Ramanujan equation is equal to

$$t^3 + pt + q = 0, \tag{32}$$

where

$$\begin{aligned}
p &= -(3\sqrt[3]{7} - 3\sqrt[3]{49} + 9), \\
q &= (6\sqrt[3]{7} - 6\sqrt[3]{49} + 2).
\end{aligned}$$

We now compute $\sum \frac{\alpha}{\beta}$ and $\sum \frac{\beta}{\alpha}$.

$$\begin{aligned}
u &= \sum \frac{\alpha}{\beta} \\
&= \sum \frac{\tan(2\delta)}{\tan(4\delta)} \\
&= \sum \frac{\tan^2(2\delta)\tan(8\delta)}{\tan(2\delta)\tan(4\delta)\tan(8\delta)} \\
&= \left(-\frac{1}{\sqrt{7}}\right)\sum \tan^2(2\delta)\tan(8\delta) \\
&= \left(-\frac{1}{\sqrt{7}}\right)\sum \tan^2(4\delta)\tan(2\delta)
\end{aligned}$$

$$
\begin{aligned}
&= \left(-\frac{1}{\sqrt{7}}\right) W_7(1,2) \\
&= \left(-\frac{1}{\sqrt{7}}\right) \left(9\sqrt{7}\right) \\
&= -9.
\end{aligned}
$$

$$
\begin{aligned}
v &= \sum \frac{\beta}{\alpha} \\
&= \sum \frac{\tan(4\delta)}{\tan(2\delta)} \\
&= \sum \frac{\tan^2(4\delta)\tan(8\delta)}{\tan(2\delta)\tan(4\delta)\tan(8\delta)} \\
&= \left(-\frac{1}{\sqrt{7}}\right) \sum \tan^2(4\delta)\tan(8\delta) \\
&= \left(-\frac{1}{\sqrt{7}}\right) \sum \tan^2(2\delta)\tan(4\delta) \\
&= \left(-\frac{1}{\sqrt{7}}\right) W_7(2,1) \\
&= \left(-\frac{1}{\sqrt{7}}\right) \left(\sqrt{7}\right) \\
&= -1.
\end{aligned}
$$

By Theorem 1.7,

$$
q^2 + \frac{4p^3}{27} = \left(\sum \frac{\alpha}{\beta} - \sum \frac{\beta}{\alpha}\right)^2 = 64.
$$

$$
\begin{aligned}
t &= \frac{1}{\sqrt[3]{2}} \sqrt[3]{-q + \sqrt{q^2 + \frac{4p^3}{27}}} + \frac{1}{\sqrt[3]{2}} \sqrt[3]{-q - \sqrt{q^2 + \frac{4p^3}{27}}} \\
&= \frac{1}{\sqrt[3]{2}} \sqrt[3]{-q + 8} + \frac{1}{\sqrt[3]{2}} \sqrt[3]{-q - 8} \\
&= \frac{1}{\sqrt[3]{2}} \sqrt[3]{(6\sqrt[3]{7} - 6\sqrt[3]{49} + 2) + 8} + \frac{1}{\sqrt[3]{2}} \sqrt[3]{(6\sqrt[3]{7} - 6\sqrt[3]{49} + 2) - 8} \\
&= \sqrt[3]{(3\sqrt[3]{7} - 3\sqrt[3]{49} + 1) + 4} + \sqrt[3]{(3\sqrt[3]{7} - 3\sqrt[3]{49} + 1) - 4} \\
&= \sqrt[3]{3\sqrt[3]{7} - 3\sqrt[3]{49} + 5} + \sqrt[3]{3\sqrt[3]{7} - 3\sqrt[3]{49} - 3}.
\end{aligned}
$$

By Theorem 1.5,

$$
\begin{aligned}
\sum \sqrt[3]{\alpha} &= \sqrt[3]{a + 6 + 3t} \\
&= \sqrt[3]{\sqrt[3]{7} + 6 + 3\left(\sqrt[3]{3\sqrt[3]{7} - 3\sqrt[3]{49} + 5} + \sqrt[3]{3\sqrt[3]{7} - 3\sqrt[3]{49} - 3}\right)}
\end{aligned}
$$

and

$$
\sum \frac{1}{\sqrt[3]{\alpha}} = \sqrt[3]{b + 6 + 3t}
$$

$$= \sqrt[3]{-\sqrt[3]{49} + 6 + 3\left(\sqrt[3]{3\sqrt[3]{7} - 3\sqrt[3]{49} + 5} + \sqrt[3]{3\sqrt[3]{7} - 3\sqrt[3]{49} - 3}\right)}.$$

This implies Equations (9) and (10) in Theorem 1.4. $\square$

8 Remarks

In a future paper, we will use the methods developed in this paper to compute Ramanujan type identities when the roots are the following forms:

$$k^{m-n}\frac{tfn_1^m}{tfn_2^n},\ \ k^{m-n}\frac{tfn_2^m}{tfn_3^n},\ \ k^{m-n}\frac{tfn_3^m}{tfn_1^n},$$

where tfn_1, tfn_2, tfn_3 are trigonometric values.

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