De Moivre-type identities
for the Jacobsthal numbers

Mücahit Akbiyik[1,*] and Seda Yamaç Akbiyik[2]

[1] Department of Mathematics, Beykent University
Büyükçekmece / Istanbul, Turkey
e-mail: `mucahitakbiyik@beykent.edu.tr`

[2] Department of Computer Engineering, Istanbul Gelisim University
Avcılar / Istanbul, Turkey
e-mail: `syamac@gelisim.edu.tr`

[*] *Corresponding author*

Abstract: The main aim of this study is to obtain De Moivre-type identities for Jacobsthal numbers. Also, this paper presents a method for constructing the second order Jacobsthal and Jacobsthal third-order numbers and the third-order Jacobsthal and Jacobsthal–Lucas numbers. Moreover, we give some interesting identities, such as Binet's formulas for some specific third-order Jacobsthal numbers that we derive from De Moivre-type identities.

Keywords: De Moivre-type identity, Jacobsthal numbers, Binet's formula.

2020 Mathematics Subject Classification: 11B39, 11B83.

1 Introduction

In the literature, the roots of the equation $x^2 - x - 1 = 0$ are given as

$$x_1 = (1 + \sqrt{5})/2,$$
$$x_2 = (1 - \sqrt{5})/2,$$

and the following relation is satisfied

$$\left(\frac{1 \pm \sqrt{5}}{2}\right)^n = \frac{L_n \pm \sqrt{5}F_n}{2}, \tag{1}$$

where L_n denotes the n-th Lucas number and F_n denotes the n-th Fibonacci number. The relation (1) is called *De Moivre-type identity* for Fibonacci numbers which is proposed by Stephen Fisk in [1, 2].

The Tribonacci and the Tetranacci numbers are like the Fibonacci numbers. But the Tribonacci sequence starts with three predetermined terms and each term afterwards is the sum of the preceding three terms instead of starting with two predetermined terms. Similarly, the Tetranacci sequence starts with four predetermined terms and each term afterwards is the sum of the preceding four terms. Lin in [10] and [11], gave the De Moivre-type identities for the Tribonacci and the Tetranacci numbers by using the equation $x^3 - x^2 - x - 1 = 0$ and the equation $x^4 - x^3 - x^2 - x - 1 = 0$, respectively.

Horadam in [8], defined the Jacobsthal sequence and he obtained some identities for the Jacobsthal numbers. Also, Cook et. al. in [7] introduced the third-order Jacobsthal numbers and presented some properties for the third-order Jacobsthal numbers. Moreover, there are a lot of articles about Jacobsthal number such as [4–9] in the literature.

In this paper, the De Moivre-type identities for Jacobsthal numbers and the third-order Jacobsthal numbers are shown by using the equation $x^2 - x - 2 = 0$ and the equation $x^3 - x^2 - x - 2 = 0$, respectively.

2 De Moivre-type identities for Jacobsthal numbers

In this section, the De Moivre-type identity is given for the Jacobsthal numbers. The roots of the equation $x^2 - x - 2 = 0$ are

$$r_1 = (1 + \sqrt{9})/2 = 2,$$
$$r_2 = (1 - \sqrt{9})/2 = -1,$$

[8]. Moreover, the roots can be written in the following form:

$$r_{1,2} = \frac{1 \pm \sqrt{9}}{2}. \tag{2}$$

The De Moivre-type identity for the Jacobsthal numbers can be found as follow:

$$\left(\frac{1 \pm \sqrt{9}}{2}\right)^n = \frac{j_n \pm 3J_n}{2}, \tag{3}$$

where j_n denotes the n-th Jacobsthal–Lucas numbers and J_n denotes the n-th Jacobsthal numbers.

Note that the sequence $\{j_n\}$ is the Jacobsthal–Lucas sequence in [8] (or the sequence A014551 in [12]) with the recurrence relation $j_n = j_{n-1} + 2j_{n-2}$, for $n > 1$ and $j_0 = 2, j_1 = 1$. Also, the sequence $\{J_n\}$ is the Jacobsthal sequence whose recurrence relation is $J_n = J_{n-1} + 2J_{n-2}$, for $n > 1$ and $J_0 = 0, J_1 = 1$, in [8].

3 De Moivre-type identities for the third-order Jacobsthal numbers

In this section, the De Moivre-type identity is given for the third-order Jacobsthal numbers. The three roots of the equation

$$x^3 - x^2 - x - 2 = 0 \tag{4}$$

are

$$r_1 = 2, \tag{5}$$

$$r_2 = \frac{-1 + i\sqrt{3}}{2}, \tag{6}$$

$$r_3 = \frac{-1 - i\sqrt{3}}{2}, \tag{7}$$

[7]. Moreover, these three roots can be written in the following forms:

$$r_1 = \frac{1}{3} + X + Y, \tag{8}$$

$$r_2 = \frac{1}{3} - \frac{1}{2}(X + Y) + i\frac{\sqrt{3}}{2}(X - Y), \tag{9}$$

$$r_3 = \frac{1}{3} - \frac{1}{2}(X + Y) - i\frac{\sqrt{3}}{2}(X - Y), \tag{10}$$

where $X = \frac{4}{3}, Y = \frac{1}{3}$. Thus, the powers of the root r_1 can be calculated under the condition $XY = \frac{4}{9}$ and $X^3 + Y^3 = \frac{65}{27}$ as follows:

$$r_1^2 = \frac{3}{3} + \frac{2}{3}(X + Y) + (X^2 + Y^2), \tag{11}$$

$$r_1^3 = \frac{10}{3} + \frac{5}{3}(X + Y) + (X^2 + Y^2), \tag{12}$$

$$r_1^4 = \frac{15}{3} + \frac{13}{3}(X + Y) + 2(X^2 + Y^2), \tag{13}$$

$$r_1^5 = \frac{31}{3} + \frac{22}{3}(X + Y) + 5(X^2 + Y^2), \tag{14}$$

$$r_1^6 = \frac{66}{3} + \frac{45}{3}(X + Y) + 9(X^2 + Y^2) \tag{15}$$

$$\vdots$$

The coefficients of the above equations construct three third-order Jacobsthal sequences which are denoted by $\{R_n\}$, $\{S_n\}$, and $\{T_n\}$, respectively. Thus,

1. the sequence $\{R_n\}$ is the third-order Jacobsthal sequence with the recurrence relation $R_n = R_{n-1} + R_{n-2} + 2R_{n-3}$ for $n \geq 3$ and $R_0 = 3, R_1 = 1, R_2 = 3$. Note that the sequence $\{R_n\}$ is called as the modified third-order Jacobsthal sequence by Cerda-Morales in [6], (it could have been considered as the third-order Jacobsthal–Lucas sequence, but this was not mentioned in [6]).

2. the sequence $\{S_n\}$ is the third-order Jacobsthal sequence with recurrence relation $S_n = S_{n-1} + S_{n-2} + 2S_{n-3}$ for $n \geq 3$ and $S_0 = 3, S_1 = 2$ and $S_2 = 5$.

3. the sequence $\{T_n\}$ is the third-order Jacobsthal sequence with the recurrence relation $T_n = T_{n-1} + T_{n-2} + 2T_{n-3}$ for $n \geq 3$ and $T_0 = 1, T_1 = 1$ and $T_2 = 2$. Note that the sequence $\{T_n\}$ is the 1-shifted sequence of the sequence defined by Cook et al. with the initial conditions $T_0 = 0, T_1 = 1$ and $T_2 = 1$ in [7].

The first 11 terms of above sequences are presented in the following Table 1.

n	0	1	2	3	4	5	6	7	8	9	10
R_n	3	1	3	10	15	31	66	127	255	514	1023
S_n	3	2	5	13	22	45	93	182	365	733	1462
T_n	1	1	2	5	9	18	37	73	146	293	585

Table 1. The third-order Jacobsthal numbers

By considering the sequences $\{R_n\}$, $\{S_n\}$, $\{T_n\}$, $XY = \frac{4}{9}$, $X^3 + Y^3 = \frac{65}{27}$, and applying induction over n, we construct the following from the equations (11)–(15):

$$r_1^n = \frac{1}{3}R_n + \frac{1}{3}S_{n-1}(X+Y) + T_{n-2}(X^2 + Y^2). \tag{16}$$

Similarly, we get

$$r_2^n = \frac{1}{3}R_n - \frac{1}{6}S_{n-1}(X+Y) - \frac{1}{2}T_{n-2}(X^2+Y^2) + \sqrt{3}i[\frac{1}{6}S_{n-1}(X-Y) - \frac{1}{2}T_{n-2}(X^2-Y^2)], \tag{17}$$

and

$$r_3^n = \frac{1}{3}R_n - \frac{1}{6}S_{n-1}(X+Y) - \frac{1}{2}T_{n-2}(X^2+Y^2) - \sqrt{3}i[\frac{1}{6}S_{n-1}(X-Y) - \frac{1}{2}T_{n-2}(X^2-Y^2)]. \tag{18}$$

Hence, the relations (16)–(18) are called De Moivre-type identities for the third-order Jacobsthal numbers.

4   Generating functions and Binet's formula for the numbers R_n, S_n, and T_n

In this section, we get the generating functions for the sequences $\{R_n\}$, $\{S_n\}$, and $\{T_n\}$, by using the recurrence relations and the initial conditions of them. Firstly, the generating function for $\{R_n\}$ can be found as

$$R(x) = \frac{3 - 2x - x^2}{1 - x - x^2 - 2x^3} \tag{19}$$

where $R(x) = \sum_{n=0}^{\infty} R_n x^n$, as in [6].

The generating functions for the sequences $\{T_n\}$ can be calculated as follows:

$$T(x) = \sum_{n=0}^{\infty} T_n x^n = 1 + x + 2x^2 + 5x^3 + \sum_{n=4}^{\infty} T_n x^n \tag{20}$$

and by substituting the recurrence relation of T_n, we have

$$T(x) = 1 + x + 2x^2 + 5x^3 + x(T(x) - 1 - x - 2x^2) + x^2(T(x) - 1 - x) + 2x^3(T(x) - 1). \tag{21}$$

So we obtain

$$T(x) = \frac{1}{1 - x - x^2 - 2x^3}.$$

Similarly, we get

$$S(x) = \frac{3 - x}{1 - x - x^2 - 2x^3}, \tag{22}$$

where $S(x) = \sum_{n=0}^{\infty} S_n x^n$.

Thus, the Binet's formulas for sequence $\{R_n\}, \{S_n\}$, and $\{T_n\}$ can be found as follows:

$$R_n = r_1^n + r_2^n + r_3^n, \tag{23}$$

$$T_n = \frac{r_1^{n+2}}{(r_1 - r_2)(r_1 - r_3)} + \frac{r_2^{n+2}}{(r_2 - r_1)(r_2 - r_3)} + \frac{r_3^{n+2}}{(r_3 - r_1)(r_3 - r_2)}, \tag{24}$$

$$S_n = \frac{(3r_1 - 1)r_1^{n+1}}{(r_1 - r_2)(r_1 - r_3)} + \frac{(3r_2 - 1)r_2^{n+1}}{(r_2 - r_1)(r_2 - r_3)} + \frac{(3r_3 - 1)r_3^{n+1}}{(r_3 - r_1)(r_3 - r_2)}. \tag{25}$$

Note that the equation (23) was obtained in [6]. Moreover, for the proof of the equation (24), we seek for constants d_1, d_2 and d_3 such that

$$T_n = d_1 r_1^n + d_2 r_2^n + d_3 r_3^n.$$

The result is found by solving the system of linear equations for $n = 0$, $n = 1$ and $n = 2$,

$$d_1 + d_2 + d_3 = 1$$
$$d_1 r_1 + d_2 r_2 + d_3 r_3 = 1$$
$$d_1 r_1^2 + d_2 r_2^2 + d_3 r_3^2 = 2.$$

Hence, we find

$$d_1 = \frac{r_1^2}{(r_1 - r_2)(r_2 - r_3)},$$
$$d_2 = \frac{r_2^2}{(r_2 - r_1)(r_2 - r_3)},$$
$$d_3 = \frac{r_3^2}{(r_3 - r_1)(r_3 - r_2)}.$$

Similarly, the Binet's formula of S_n can be obtained as in equation (25).

5   Some properties of R_n, S_n, and T_n

In this section, some interesting identities are derived by using the definitions of the third-order Jacobsthal numbers. We know that the recurrence relations of the sequences $\{R_n\}$, $\{S_n\}$ and $\{T_n\}$ are follows:

$$R_{n+3} = R_{n+2} + R_{n+1} + 2R_n, \quad \text{for } n \geq 3, R_0 = 3, R_1 = 1, R_2 = 3,$$
$$S_{n+3} = S_{n+2} + S_{n+1} + 2S_n, \quad \text{for } n \geq 3, S_0 = 3, S_1 = 2, S_2 = 5,$$
$$T_{n+3} = T_{n+2} + T_{n+1} + 2T_n, \quad \text{for } n \geq 3, T_0 = 1, T_1 = 1, T_2 = 2.$$

By using these relations, we can write

$$
\begin{aligned}
T_0 &= 1, \\
T_1 &= 1, \\
T_2 &= 2, \\
T_3 &= 2 + 3 = 5 = 2T_2 + 1, \\
T_4 &= 5 + 4 = 9 = 2T_3 - 1, \\
T_5 &= 9 + 9 = 18 = 2T_4, \\
T_6 &= 18 + 19 = 37 = 2T_5 + 1, \\
T_7 &= 37 + 36 = 73 = 2T_6 - 1, \\
T_8 &= 73 + 73 = 146 = 2T_7, \\
T_9 &= 146 + 147 = 293 = 2T_8 + 1, \\
&\vdots
\end{aligned}
$$

So, it is seen that we have

$$
T_n = \begin{cases}
2T_{n-1} + 1, & n \equiv 0 \mod 3 \\
2T_{n-1} - 1, & n \equiv 1 \mod 3 \text{ for } n \geq 3. \\
2T_{n-1}, & n \equiv 2 \mod 3
\end{cases}
\tag{26}
$$

Similarly, we can find

$$
S_n = \begin{cases}
2S_{n-1} + 3, & n \equiv 0 \mod 3 \\
2S_{n-1} - 4, & n \equiv 1 \mod 3 \text{ for } n \geq 3, \\
2S_{n-1} + 1, & n \equiv 2 \mod 3
\end{cases}
\tag{27}
$$

and

$$
R_n = \begin{cases}
2R_{n-1} + 4, & n \equiv 0 \mod 3 \\
2R_{n-1} - 5, & n \equiv 1 \mod 3 \text{ for } n \geq 3. \\
2R_{n-1} + 1, & n \equiv 2 \mod 3
\end{cases}
\tag{28}
$$

Moreover, from the definition T_n and S_n, we have

$$S_n = 3T_n - T_{n-1}, \text{ for } n \geq 1.$$

Now, we define a new sequence $\{U_n\}$ with the recurrence relation

$$U_n = T_{n-1} + 2T_{n-2}, \text{ for } n \geq 2,$$

and $U_0 = 0, U_1 = 1$. Note that the sequence $\{U_n\}$ is also a third-order Jacobsthal sequence with the recurrence relation

$$U_{n+3} = U_{n+2} + U_{n+1} + 2U_n,$$

for $n \geq 3$ and $U_0 = 0, U_1 = 1, U_2 = 3$. The first some terms of the sequence U_n is

$$0, 1, 3, 4, 9, 19, 36, 73, 147, 292, 585, \ldots$$

Also, the sequence has the similar definition in the articles [10] and [3]. Hence, from the definitions of the sequences $\{R_n\}, \{S_n\}, \{T_n\}$ and $\{U_n\}$ and similar to equation (26), we can calculate the three following equations:

$$T_n = \begin{cases} U_n + 1, & n \equiv 0 \mod 3 \\ U_n, & n \equiv 1 \mod 3 \quad \text{for } n \geq 3, \\ U_n - 1, & n \equiv 2 \mod 3 \end{cases} \tag{29}$$

$$S_n = \begin{cases} 3U_n - U_{n-1} + 4, & n \equiv 0 \mod 3 \\ 3U_n - U_{n-1} - 1, & n \equiv 1 \mod 3 \quad \text{for } n \geq 1, \\ 3U_n - U_{n-1} - 3, & n \equiv 2 \mod 3 \end{cases} \tag{30}$$

and

$$R_n = 3U_{n-1} + U_{n-2}, \text{ for } n \geq 2. \tag{31}$$

By using the recurrence relation of R_n, we have the following linear equation system:

$$\begin{aligned} R_3 &= R_2 + R_1 + 2R_0, \\ R_4 &= R_3 + R_2 + 2R_1, \\ &\vdots \\ R_{n+3} &= R_{n+2} + R_{n+1} + 2R_n. \end{aligned}$$

So, we can write

$$\sum_{k=3}^{n+3} R_k = \sum_{k=2}^{n+2} R_k + \sum_{k=1}^{n+1} R_k + 2\sum_{k=0}^{n} R_k.$$

Furthermore, from the recurrence relations of R_n and U_n, we can write

$$R_{n+3} + R_{n+2} + R_{n+1} - 3 - 1 - 3 + \sum_{k=0}^{n} R_k = R_{n+2} + R_{n+1} - 3 - 1 + \sum_{k=0}^{n} R_k + R_{n+1} - 3$$

$$+ \sum_{k=0}^{n} R_k + 2\sum_{k=0}^{n} R_k.$$

So, we have the following summation formula

$$\sum_{k=0}^{n} R_k = \frac{4U_{n+1} + 5U_{n-1}}{3}. \tag{32}$$

Similarly, we can calculate

$$\sum_{k=0}^{n} S_k = \frac{S_{n+3} - S_{n+1} - 2}{3} \tag{33}$$

$$= \frac{T_{n+2} + 7T_n - 2}{3} \tag{34}$$

and

$$\sum_{k=0}^{n} U_k = \frac{U_{n+3} - U_{n+1} - 3}{3}. \tag{35}$$

Finally, we know that

$$T_0 T_1 + T_1 T_2 + \cdots + T_{n-1} T_n = \sum_{k=1}^{n} T_{k-1} T_k \tag{36}$$

is satisfied. Also, if the equation (29) is substituted in equation (36), we obtain for $n \geq 3$

$$\sum_{k=1}^{n} T_{k-1} T_k = \begin{cases} \sum_{k=1}^{n} (U_{k-1} U_k) + \sum_{k=1}^{n/3} (U_{3k-1} - U_{3k} - 1), & n \equiv 0 \mod 3 \\ \sum_{k=1}^{n} (U_{k-1} U_k) + \sum_{k=1}^{(n-1)/3} (U_{3k-1} - U_{3k} - 1) + U_n, & n \equiv 1 \mod 3 \\ \sum_{k=1}^{n} (U_{k-1} U_k) + \sum_{k=1}^{n-2/3} (U_{3k-1} - U_{3k} - 1), & n \equiv 2 \mod 3. \end{cases}$$

6 Conclusion

In this article, we obtain the De Moivre-type identities for the Jacobsthal sequences. In the future, this identity can be derived for the fourth-order Jacobsthal numbers, similar to the De Moivre-type identity for Tetranacci numbers in [11].

Acknowledgements

References

[1] Basin, S. L. (ed.) (1963). Elementary problems and solutions. *The Fibonacci Quarterly*, 1(2), 85.

[2] Basin, S. L. (ed.) (1963). Elementary problems and solutions. *The Fibonacci Quarterly*, 1(4), 77.

[3] Bruce, I. (1984). A modified Tribonacci sequence. *The Fibonacci Quarterly*, 22(3), 244–246.

[4] Cerda-Morales, G. (2017). Identities for the third-order Jacobsthal quaternions. *Advances in Applied Clifford Algebras*, 27, 1043–1053.

[5] Cerda-Morales, G. (2019). On the third-order Jacobsthal and third-order Jacobsthal–Lucas sequences and their matrix representation. *Mediterranean Journal of Mathematics*, 16, Article No. 32.

[6] Cerda-Morales, G. (2020). A note on modified third-order Jacobsthal numbers. *Proyecciones Journal of Mathematics*, 39(2), 409–420.

[7] Cook, C. K., & Bacon, M. R. (2013). Some identities for Jacobsthal and Jacobsthal–Lucas numbers satisfying higher order recurrence relations. *Annales Mathematicae et Informaticae*, 41, 27—39.

[8] Horadam, A. F. (1996). Jacobsthal representation numbers. *The Fibonacci Quarterly*, 34, 40–54.

[9] Horadam, A. F. (1988). Jacobsthal and Pell curves. *The Fibonacci Quarterly*, 26, 79–83.

[10] Lin, P.-Y. (1988). De Moivre-type identities for the Tribonacci numbers. *The Fibonacci Quarterly*, 26(2), 131—134.

[11] Lin, P.-Y. (1991). De Moivre-type identities for the Tetranacci numbers. In: Bergum, G. E., Philippou, A. N., & Horadam, A. F. (eds.) *Applications of Fibonacci Numbers*. Springer, Dordrecht. DOI: 10.1007/978-94-011-3586-324.

[12] Sloane, N. J. A. (2021). *On-line Encyclopedia of Integer Sequences*. Available online at: http://oeis.org/.