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On some 3×3 dimensional matrices associated with generalized Fibonacci numbers

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Abstract: In this work, it is presented a procedure for finding some 3×3 dimensional matrices whose integer powers can be characterized by generalized Fibonacci numbers. Moreover, some numerical examples are given to exemplify the procedure established.
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1 Introduction and Preliminaries

It is a well known fact that Fibonacci numbers have many uses in applied sciences and real life. Many examples of the Fibonacci numbers or sequence, which lead to the golden ratio, such as rabbit problems, growth of the branches and leaves of some plants, the family tree of bees, etc., can be seen in the literature (see, for instance, [7, 8, 11]). There are also many interesting mathematical properties related to the Fibonacci sequence, the Lucas sequence and golden ratio (see, for instance, [7, 9, 11, 12]). Noteworthy, many of these features are used in mathematical modeling of some real life problems.

Now, we will recall some terminology and results to be used throughout the rest of the work.

The Fibonacci sequence $\{F_n\}$ is defined by the recurrence relation $F_{n+1} = F_n + F_{n-1}$ for any integer $n \geq 1$ together with the initial conditions $F_0 = 0$ and $F_1 = 1$. The elements of this sequence are called Fibonacci numbers [7]. There are several definitions of sequences called generalized Fibonacci sequences in the literature (see, for instance, [4, 5, 7, 10]). However, we will proceed based on the following definition.

The generalized Fibonacci sequence $\{U_n\}$ is defined by the recurrence relation $U_{n+1} = kU_n + tU_{n-1}$ for any integer $n \geq 1$ and the initial conditions $U_0 = 0$, $U_1 = 1$ with k and t being nonzero real numbers [10]. The elements of the sequence $\{U_n\}$ are called generalized Fibonacci numbers.

Note that the Fibonacci sequence $\{F_n\}$ is a special version of the generalized Fibonacci sequence considered here with $k = t = 1$.

The roots of the equation $x^2 - kx - t = 0$ are $\alpha_{k,t} = \frac{k + \sqrt{k^2 + 4t}}{2}$ and $\beta_{k,t} = \frac{k - \sqrt{k^2 + 4t}}{2}$. The relation

$$U_n = \frac{\alpha_{k,t}^n - \beta_{k,t}^n}{\alpha_{k,t} - \beta_{k,t}}$$

is known as Binet's formula with $k^2 + 4t > 0$ and $n \in \mathbb{Z}$. In addition, for $n \in \mathbb{Z}$, the identities

$$\alpha_{k,t}^n = \alpha_{k,t} U_n + tU_{n-1} \quad \text{and} \quad \beta_{k,t}^n = \beta_{k,t} U_n + tU_{n-1} \tag{1}$$

hold. Also, negatively indexed generalized Fibonacci numbers are given by the relation $U_{-n} = \frac{-U_n}{(-t)^n}$ [10].

It is well known that there are some relations between Fibonacci numbers or generalized Fibonacci numbers and matrices of dimension 2×2 [1–3, 7, 10]. In the study [6], a procedure that gives some relations between 3×3 dimensional matrices with eigenvalues $\alpha = \frac{1 + \sqrt{5}}{2}$, $\beta = \frac{1 - \sqrt{5}}{2}$, and 0, and Fibonacci numbers was introduced.

Inspired by the work [6], in this study, a similar procedure to that in [6] giving some relations between 3×3 dimensional matrices having the eigenvalues $\alpha_{k,t} = \frac{k + \sqrt{k^2 + 4t}}{2}$, $\beta_{k,t} = \frac{k - \sqrt{k^2 + 4t}}{2}$, and any real number r, and generalized Fibonacci numbers is presented.

2 Main result

Let the matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

be any matrix having the eigenvalues $\lambda_1 = \alpha_{k,t}$, $\lambda_2 = \beta_{k,t}$, and $\lambda_3 = r$, where r is any real number. It is obvious that the necessary and sufficient condition for the eigenvalues of A to be $\lambda_1 = \alpha_{k,t}$, $\lambda_2 = \beta_{k,t}$ and $\lambda_3 = r$ is that the equalities

$$\begin{aligned} a+e+i &= r+k \\ -ae-ai-ei+gc+hf+bd &= -kr+t \\ aei+dhc+gbf-gce-hfa-bdi &= -tr \end{aligned} \tag{2}$$

are satisfied. It can be easily proved that the conditions

$$\begin{aligned} k^2+4t &> 0 \\ r^2-kr-t &\neq 0 \end{aligned} \tag{3}$$

must hold in order for the eigenvalues $\alpha_{k,t}$, $\beta_{k,t}$ and r to be mutually different.

From now on, we will deal with 3×3 dimensional matrices that provide (2) and (3), that is, 3×3 dimensional matrices having the mutually different eigenvalues $\lambda_1=\alpha_{k,t}$, $\lambda_2=\beta_{k,t}$, and r.

Let A be a matrix satisfying all the conditions mentioned above. Moreover, let the eigenvectors corresponding to the eigenvalues $\alpha_{k,t}$, $\beta_{k,t}$ and $\lambda_3=r$ of the matrix A be $\mathbf{x}=(x_1,x_2,x_3)$, $\mathbf{y}=(y_1,y_2,y_3)$, and $\mathbf{z}=(z_1,z_2,z_3)$, respectively.

Since all eigenvalues are mutually different, the matrix A provides the equality

$$A^n = P\begin{pmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & r^n \end{pmatrix}P^{-1}$$

for all positive integers n, where P is a matrix having the columns $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$, respectively. Taking (1) into account, we get, without loss of generality,

$$\begin{aligned} A^n &= P\begin{pmatrix} U_n\lambda_1+tU_{n-1} & 0 & 0 \\ 0 & U_n\lambda_2+tU_{n-1} & 0 \\ 0 & 0 & r^n \end{pmatrix}P^{-1} \\ &= P(U_n\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & r \end{pmatrix}+tU_{n-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}+\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & r^n-rU_n-tU_{n-1} \end{pmatrix})P^{-1} \end{aligned}$$

or equivalently,

$$A^n = U_nA+tU_{n-1}I+(r^n-rU_n-tU_{n-1})D, \tag{4}$$

where $D=P\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}P^{-1}$. Taking the definition of P into consideration, we obtain

$$D=\frac{1}{\det(P)}\begin{pmatrix} z_1(x_2y_3-y_2x_3) & z_1(y_1x_3-x_1y_3) & z_1(x_1y_2-y_1x_2) \\ z_2(x_2y_3-y_2x_3) & z_2(y_1x_3-x_1y_3) & z_2(x_1y_2-y_1x_2) \\ z_3(x_2y_3-y_2x_3) & z_3(y_1x_3-x_1y_3) & z_3(x_1y_2-y_1x_2) \end{pmatrix}.$$

Here it is obvious that

$$\det(P)=z_1(x_2y_3-y_2x_3)+z_2(y_1x_3-x_1y_3)+z_3(x_1y_2-y_1x_2).$$

The system (2) has either no solution or infinitely many solutions. Since our main aim is to present a procedure to find some specific solutions to (2), we will only proceed under some certain conditions instead of characterizing all solutions.

First, let us assume that

$$x_2y_3 - y_2x_3 = y_1x_3 - x_1y_3 = x_1y_2 - y_1x_2 \tag{5}$$

and

$$z_1 + z_2 + z_3 = j \neq 0. \tag{6}$$

In this case, we have $\det(P) = j(x_2y_3 - y_2x_3)$ and $D = \frac{1}{j}\begin{pmatrix} z_1 & z_1 & z_1 \\ z_2 & z_2 & z_2 \\ z_3 & z_3 & z_3 \end{pmatrix}$. Thus, we need to find a matrix A satisfying the equalities (2), (5), and (6), and the inequalities (3) simultaneously. Now let us say that, without loss of generality, $x_2y_3 - y_2x_3 = y_1x_3 - x_1y_3 = x_1y_2 - y_1x_2 = s$. Note that $s \neq 0$ since $\det(P) \neq 0$. Also, since the cross product vector $\mathbf{x} \times \mathbf{y}$ is perpendicular to both the vectors $\mathbf{x}$ and $\mathbf{y}$, the equalities

$$x_1 + x_2 + x_3 = y_1 + y_2 + y_3 = 0 \tag{7}$$

are obtained.

Thus, (5) leads to (7). It is easily seen that the converse of this conclusion is also true. Thus, (5) and (7) are equivalent. From now on, we will consider (7) instead of (5) for simplicity.

Next, let us proceed as follows. According to the hypotheses, we have the systems

$$A\mathbf{x} = \lambda_1\mathbf{x} \tag{8}$$

and

$$A\mathbf{y} = \lambda_2\mathbf{y}. \tag{9}$$

Taking (7) into account, from (8) and (9), we get

$$(a + d + g - c - f - i)x_1 + (b + e + h - c - f - i)x_2 = 0$$

and

$$(a + d + g - c - f - i)y_1 + (b + e + h - c - f - i)y_2 = 0,$$

respectively. For the sake of simplicity, putting $a+d+g-c-f-i = c_1$ and $b+e+h-c-f-i = c_2$, the system $\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is obtained. Hence, it is seen that $c_1 = c_2 = 0$; that is $a + d + g - c - f - i = 0$ and $b + e + h - c - f - i = 0$. So, we get

$$a + d + g = c + f + i = b + e + h \tag{10}$$

under the hypotheses. Meanwhile, note that we have also the system

$$A\mathbf{z} = r\mathbf{z}. \tag{11}$$

If we consider (10), then from the systems (8) and (9), we obtain

$$\begin{aligned}(a+d+g)(x_1+x_2+x_3) &= \lambda_1(x_1+x_2+x_3),\\ (a+d+g)(y_1+y_2+y_3) &= \lambda_2(y_1+y_2+y_3),\end{aligned} \tag{12}$$

respectively. And from the equalities (12), we get $x_1 + x_2 + x_3 = y_1 + y_2 + y_3 = 0$. Otherwise, the following contradictions are obtained in each case: If $x_1 + x_2 + x_3 \neq 0$ and $y_1 + y_2 + y_3 \neq 0$, then the equalities (12) lead to the contradiction $\lambda_1 = \lambda_2$. Taking (6) and (10) into account, if $x_1 + x_2 + x_3 \neq 0$ and $y_1 + y_2 + y_3 = 0$, or $x_1 + x_2 + x_3 = 0$ and $y_1 + y_2 + y_3 \neq 0$, then the equalities (12) lead to the contradictions $\lambda_1 = r$, or $\lambda_2 = r$, respectively. Thus, it is obtained

$$x_1 + x_2 + x_3 = y_1 + y_2 + y_3 = 0$$

under the hypotheses.

Now, taking the equalities (6) and (10) into account, we get $a+d+g = r$ from (11). Therefore, it is now necessary to determine the matrix A such that $a + d + g = r$ under existing restrictions. So, under all these hypotheses, we have to find matrices having the from

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ r-a-d & r-b-e & r-c-f \end{pmatrix}. \tag{13}$$

If this matrix is first substituted into (11) and then the necessary elementary operations are performed, then it is obtained the system

$$\begin{aligned}(a-r)z_1 + bz_2 &= -cz_3\\ dz_1 + (e-r)z_2 &= -fz_3.\end{aligned} \tag{14}$$

Now, we will proceed as follows. Let us assume that $(a - r)(e - r) - bd \neq 0$. In this case, the general solutions of system (14) (or, equivalently (11)) are obtained as

$$z = u(-\frac{fb-(e-r)c}{bd-(a-r)(e-r)},\ -\frac{cd-(a-r)f}{bd-(a-r)(e-r)},\ 1)$$

with $u \neq 0$. Hence, taking (6) into account, we get

$$u(-\frac{fb-(e-r)c}{bd-(a-r)(e-r)} - \frac{cd-(a-r)f}{bd-(a-r)(e-r)} + 1) = j.$$

If u is chosen as $u = j$, then it is obtained that $fb-(e-r)c+cd-(a-r)f = 0$. Thus, we have included the additional conditions

$$f(b-a+r) + c(d-e+r) = 0 \quad \text{and} \quad bd-(a-r)(e-r) \neq 0. \tag{15}$$

Next, let us reconsider the equalities (2) and (3) under all hypotheses so far. Considering (13) and the first equality in (15), it seen that the equations (2) turn into the equations

$$\begin{aligned}a-c-f+e &= k,\\ -ae-ar-er+2rc+2rf+bd &= -kr+t,\\ fr+cr-ae+bd &= t,\end{aligned}$$

or equivalently, the equations

$$\begin{aligned} a - c - f + e &= k \\ fr + cr - ae + bd &= t. \end{aligned}$$

Thus, we must have the following:

$$\begin{aligned} f(b - a + r) + c(d - e + r) &= 0, \\ bd - (a - r)(e - r) &\neq 0, \\ a - c - f + e &= k, \\ fr + cr - ae + bd &= t. \end{aligned} \tag{16}$$

If the conditions in (3) are added to those in (16), then we obtain the conditions that will satisfy all hypotheses assumed so far as follows;

$$\begin{aligned} k^2 + 4t &> 0, \\ r^2 - kr - t &\neq 0, \\ f(b - a + r) + c(d - e + r) &= 0, \\ bd - (a - r)(e - r) &\neq 0, \\ a - c - f + e &= k, \\ fr + cr - ae + bd &= t, \end{aligned}$$

or equivalently,

$$\begin{aligned} k^2 + 4t &> 0, \\ r^2 - kr - t &\neq 0, \\ f(b - a + r) + c(d - e + r) &= 0, \\ a - c - f + e &= k, \\ fr + cr - ae + bd &= t. \end{aligned} \tag{17}$$

Consequently, we have shown that any matrix A in the form (13) satisfying all the conditions in (17) has eigenvalues $\alpha_{k,t}$, $\beta_{k,t}$, and r, which are mutually different.

The relation given by (4) between generalized Fibonacci numbers and any positive integer power of any matrix A satisfying all the conditions in (17) will, generally, have a roughness. In fact, our main aim is to present a procedure for the problem considered. So, we want to close the study by giving a result under a specific choice in order to obtain a descriptive result related to the procedure presented to the problem handled in the study.

For example, let us choose $c = f = 0$. In this case, the equalities (17) turn into the equalities

$$\begin{aligned} k^2 + 4t &> 0, \\ r^2 - kr - t &\neq 0, \\ a + e &= k, \\ -ae + bd &= t. \end{aligned}$$

Moreover, the matrix A of the form (13) is also obtained as

$$A = \begin{pmatrix} a & b & 0 \\ \dfrac{ak - a^2 + t}{b} & k - a & 0 \\ \dfrac{br - ab - ak + a^2 - t}{b} & r + a - b - k & r \end{pmatrix}.$$

Taking the equality (4) into account with

$$D = \frac{1}{\det(P)} \begin{pmatrix} z_1 & z_1 & z_1 \\ z_2 & z_2 & z_2 \\ z_3 & z_3 & z_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix},$$

it is obtained that

$$A^n = U_n \begin{pmatrix} a & b & 0 \\ \dfrac{ak - a^2 + t}{b} & k - a & 0 \\ \dfrac{br - ab - ak + a^2 - t}{b} & r + a - b - k & r \end{pmatrix} + tU_{n-1} I + (r^n - rU_n - tU_{n-1}) D,$$

or equivalently,

$$A^n = \begin{pmatrix} aU_n + tU_{n-1} & bU_n & 0 \\ \dfrac{ak - a^2 + t}{b} U_n & (k - a)U_n + tU_{n-1} & 0 \\ \dfrac{-ab - ak + a^2 - t}{b} U_n - tU_{n-1} + r^n & (a - b - k)U_n - tU_{n-1} + r^n & r^n \end{pmatrix}$$

for all positive integers n.

Thus, the following result has been proved.

Theorem 2.1. *The eigenvalues of the matrix*

$$A = \begin{pmatrix} a & b & 0 \\ \dfrac{ak - a^2 + t}{b} & k - a & 0 \\ \dfrac{br - ab - ak + a^2 - t}{b} & r + a - b - k & r \end{pmatrix}$$

are $\alpha_{k,t}$, $\beta_{k,t}$, and r, where a, b, k, and t are any real numbers provided that b, k, and t are not zero, $k^2 + 4t > 0$, and $r^2 - kr - t \neq 0$. Moreover,

$$A^n = \begin{pmatrix} aU_n + tU_{n-1} & bU_n & 0 \\ \dfrac{ak - a^2 + t}{b} U_n & (k - a)U_n + tU_{n-1} & 0 \\ \dfrac{-ab - ak + a^2 - t}{b} U_n - tU_{n-1} + r^n & (a - b - k)U_n - tU_{n-1} + r^n & r^n \end{pmatrix}$$

for all positive integers n.

If k and t in the Theorem 2.1 are chosen as $k = t = 1$, then the following result is obtained.

Corollary 2.1.1. *The eigenvalues of the matrix*

$$A = \begin{pmatrix} a & b & 0 \\ \dfrac{a-a^2+1}{b} & 1-a & 0 \\ \dfrac{br-ab-a+a^2-1}{b} & r+a-b-1 & r \end{pmatrix}$$

are $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$, *and* r, *where* a, b, r *are any real numbers with* $b \neq 0$ *and* $r^2 - r - 1 \neq 0$. *Moreover,*

$$A^n = \begin{pmatrix} aF_n + F_{n-1} & bF_n & 0 \\ \dfrac{a-a^2+1}{b}F_n & (1-a)F_n + F_{n-1} & 0 \\ \dfrac{-ab-a+a^2-1}{b}F_n - F_{n-1} + r^n & (a-b-1)F_n - F_{n-1} + r^n & r^n \end{pmatrix}$$

for all positive integers n.

Now, we give some numerical examples of matrices whose entries are associated with generalized Fibonacci numbers.

Example 2.2. *Let* $a = 1$, $b = -1$, $r = 2$, $k = 1$, *and* $t = 3$. *By Theorem 2.1, the matrix* A *becomes as in the following:*

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -3 & 0 & 0 \\ 4 & 3 & 2 \end{pmatrix}.$$

Suppose that $n = 2$. *So, by Theorem 2.1, we get*

$$A^2 = \begin{pmatrix} U_2 + 3U_1 & -U_2 & 0 \\ -3U_2 & 3U_1 & 0 \\ 2U_2 - 3U_1 + 4 & U_2 - 3U_1 + 4 & 4 \end{pmatrix} = \begin{pmatrix} 4 & -1 & 0 \\ -3 & 3 & 0 \\ 3 & 2 & 4 \end{pmatrix}.$$

Remark. Noteworthy that Theorem 2.1 is also true for all integers n in the case $r \neq 0$. For example, if $a = 2$, $b = 1$, $r = 1$, and $k = t = 1$, then

$$A = \begin{pmatrix} 2 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

and

$$A^n = \begin{pmatrix} F_{n+2} & F_n & 0 \\ -F_n & -F_{n-2} & 0 \\ -F_{n+1} + 1 & -F_{n-1} + 1 & 1 \end{pmatrix}$$

for all integers n.

Example 2.3. *Let the matrix A be as in Example 2.2. Now, suppose that $n = -3$. So, by Theorem 2.1, we get*

$$A^{-3} = \begin{pmatrix} U_{-3} + 3U_{-4} & -U_{-3} & 0 \\ -3U_{-3} & 3U_{-4} & 0 \\ 2U_{-3} - 3U_{-4} + 2^{-3} & U_{-3} - 3U_{-4} + 2^{-3} & 2^{-3} \end{pmatrix} = \begin{pmatrix} \frac{-1}{9} & \frac{-4}{27} & 0 \\ \frac{-4}{9} & \frac{-7}{27} & 0 \\ \frac{49}{72} & \frac{115}{216} & \frac{1}{8} \end{pmatrix}.$$

3 Conclusion

It has been given here a procedure to find some 3×3 dimensional matrices in the special case by starting from the problem considered in the general situation. The reason for the fact that the general situation has a roughness is that a system of linear equations has either trivial solution or infinitely many solutions. As a result, we especially want to note again that the main point of this work is to present a procedure to the problem considered in the work, or similar ones.

Also, note that the matrices in this work can also be used to develop algorithms that generate generalized Fibonacci numbers (see for instance, [13]). However, this can be considered as a new problem in itself.

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