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Note on translated sum on primitive sequences

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Abstract: In this note, we construct a new set $\boldsymbol{S}$ of primitive sets such that for any real number $x \geq 60$ we get:

$$\sum_{a\in\mathcal{A}} \frac{1}{a(\log a + x)} > \sum_{p\in\mathcal{P}} \frac{1}{p(\log p + x)},\ \mathcal{A} \in \boldsymbol{S},$$

where $\mathcal{P}$ denotes the set of prime numbers.

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1 Introduction

A sequence $\mathcal{A}$ of strictly positive integers is said to be primitive if none of its elements divide another. From the sequence of prime numbers $\mathcal{P} = (p_n)_{n\geq 1}$ we can construct an infinite collection of primitive sequences. Indeed, all the following sequences

$$\begin{aligned} \mathcal{A}_d^k &= \{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \mid \alpha_1, \ldots, \alpha_k, d \in \mathbb{N},\ \alpha_1 + \cdots + \alpha_k = d,\ d \geq 1\}, \\ \mathcal{A}^k &= \{p_n \in \mathcal{P} \mid n > k\},\ \mathcal{B}_d^k = \mathcal{A}_d^k \cup \mathcal{A}^k, \end{aligned}$$

are primitive. According to the prime number theorem, the n-th prime number p_n is asymptotically equal to $n \log n$; this ensures the convergence of the series

$$S(\mathcal{P}) = \sum_{p\in\mathcal{P}} \frac{1}{p \log p}.$$

A computation for $S(\mathcal{P})$ was obtained in [2] by Cohen as:

$$S(\mathcal{P}) = 1.636616323351260868569658003921863671181597076131 29\ldots\ .$$

Throughout this paper we assume that $\mathcal{A} \notin \{\varnothing, \{1\}\}$. In [3], Erdős proved that the series $S(\mathcal{A})$ converges for any primitive sequence $\mathcal{A}$ and in [4], Erdős asked if it is true that $S(\mathcal{A}) \leq S(\mathcal{P})$ for any primitive sequence $\mathcal{A}$. In [5], Erdős and Zhang showed that $S(\mathcal{A}) \leq 1.84$ for any primitive sequence $\mathcal{A}$, and in [1], Clark improved this result $S(\mathcal{A}) \leq e^{\gamma}$ (where γ is the Euler constant) in the special case when $\mathcal{A}$ is a primitive set of composite numbers. Several years later in [8], Lichtman and Pomerance proved that $S(\mathcal{A}) < e^{\gamma} \simeq 1.781$. Moreover, in [5], Erdős conjectured that $S(\mathcal{A}) \leq S(\mathcal{P})$ for any primitive sequence $\mathcal{A}$, then in [11, 12], Zhang proved this conjecture in some special cases of primitive sequences. In [7], the authors show that the analogue of the Erdős conjecture, which was studied by Farhi in [6], is not satisfied for the translated sums of the form:

$$S(\mathcal{A}, x) = \sum_{a \in \mathcal{A}} \frac{1}{a(\log a + x)} \text{ for } x \geq 81.$$

Later in [9], the authors show that for x large enough, there exists a primitive sequence $\mathcal{A}$, such that $S(\mathcal{A}, x) >> S(\mathcal{P}, x)$. In this article, we improve the result of [7] as follows:

Theorem 1.1. *Let $k_0 = 130947$ and $x_0 = 60$. For any integer $k \geq k_0$, we have:*

$$S(\mathcal{B}_2^k, x) > S(\mathcal{P}, x) \text{ for } x \geq x_0.$$

To prove this theorem, we need the following lemmas.

2 Lemmas

Lemma 2.1. ([10]) For any real number $x > 1$, we have:

$$\sum_{p \in \mathcal{P}, p \leq x} \frac{1}{p} > \log\log x + \beta - \frac{1}{2\log^2 x} \text{ where } \beta = 0.261497212847643\ldots$$

Lemma 2.2. *Let $x_1, x_1, \ldots, x_n$ be non-zero real numbers, then we have:*

$$\sum_{1 \leq i \leq j \leq n} x_i x_j = \frac{1}{2}\left(\left(\sum_{1 \leq i \leq n} x_i\right)^2 + \sum_{1 \leq i \leq n} x_i^2\right). \tag{1}$$

Proof. By induction. □

Lemma 2.3. ([7]) For all integer $k \geq 1$ and all integer $d \geq 2$, we have the (disjoint) union

$$\mathcal{A}_d^{k+1} = \mathcal{A}_d^k \cup \left\{ap_{k+1} | a \in \mathcal{A}_{d-1}^{k+1}\right\}.$$

Lemma 2.4. *Let $k' = 58$. For any real number $x > 0$, the sequence $\left(\mathcal{S}(\mathcal{B}_2^k, x)\right)_{k \geq k'}$ is strictly increasing.*

Proof. According to Lemma 2.2, for any integer $k \geq 1$ we have:

$$\sum_{a \in \mathcal{A}_2^k} \frac{1}{a} = \sum_{1 \leq i \leq j \leq k} \frac{1}{p_i p_j} = \frac{1}{2}\left(\left(\sum_{1 \leq i \leq k} \frac{1}{p_i}\right)^2 + \sum_{1 \leq i \leq k} \left(\frac{1}{p_i}\right)^2\right).$$

For $1 \le i \le k$, we have $p_i \le p_k$, so

$$\begin{aligned}\sum_{a \in \mathcal{A}_2^k} \frac{1}{a} &\ge \frac{1}{2}\left(\left(\sum_{1\le i\le k} \frac{1}{p_i}\right)^2 + \frac{1}{p_k}\sum_{1\le i\le k}\frac{1}{p_i}\right)\\ &\ge \frac{1}{2}\left(\sum_{1\le i\le k}\frac{1}{p_i} + \frac{1}{p_k}\right)\sum_{1\le i\le k}\frac{1}{p_i}.\end{aligned}$$

According to Lemma 2.3, we have:

$$\mathcal{B}_2^{k+1} = \mathcal{A}_2^{k+1} \cup \mathcal{A}^{k+1} = \mathcal{A}_2^k \cup \left\{ap_{k+1} | a \in \mathcal{A}_1^{k+1}\right\} \cup \mathcal{A}^{k+1},$$

so

$$S\left(\mathcal{B}_2^{k+1}, x\right) - S\left(\mathcal{B}_2^k, x\right) = \frac{1}{p_{k+1}}\left(S\left(\mathcal{A}_1^{k+1}, \log p_{k+1} + x\right) - \frac{1}{\log p_{k+1} + x}\right).$$

Knowing that p_{k+1} is the largest element of $\mathcal{A}_1^{k+1}$, so we have:

$$S\left(\mathcal{A}_1^{k+1}, \log p_{k+1} + x\right) = \sum_{a\in\mathcal{A}_1^{k+1}} \frac{1}{a(\log a + \log p_{k+1} + x)} \ge \frac{1}{2\log p_{k+1} + x}\sum_{n=1}^{k+1}\frac{1}{p_n}.$$

A computer calculation gives

$$\sum_{n=1}^{k+1}\frac{1}{p_n} \ge \sum_{n=1}^{k'+1}\frac{1}{p_n} = 2.0023\ldots > 2 \text{ for } k \ge k',$$

therefore,

$$\begin{aligned}S\left(\mathcal{A}_1^{k+1}, \log p_{k+1} + x\right) - \frac{1}{\log p_{k+1} + x} &> \frac{2}{2\log p_{k+1} + x} - \frac{1}{\log p_{k+1} + x}\\ &= \frac{x}{\left(2\log p_{k+1} + x\right)\left(\log p_{k+1} + x\right)} > 0,\end{aligned}$$

so $S\left(\mathcal{B}_2^{k+1}, x\right) - S\left(\mathcal{B}_2^k, x\right) > 0$. □

3 Proof of Theorem 1.1

For any integer $k \ge 1$, the number p_k^2 is the largest element of the primitive sequence $\mathcal{A}_2^k$, so for any $a \in \mathcal{A}_2^k$ we have $\log a \le 2\log p_k$. Then for any $x > 0$, we have:

$$\begin{aligned}\sum_{a\in\mathcal{B}_2^k}\frac{1}{a(\log a + x)} &= \sum_{a\in\mathcal{A}_2^k\cup\mathcal{A}^k}\frac{1}{a(\log a + x)} = \sum_{a\in\mathcal{A}_2^k}\frac{1}{a(\log a + x)} + \sum_{a\in\mathcal{A}^k}\frac{1}{a(\log a + x)}\\ &\ge \frac{1}{2\log p_k + x}\sum_{a\in\mathcal{A}_2^k}\frac{1}{a} + \sum_{n>k}\frac{1}{p_n(\log p_n + x)}.\end{aligned}$$

From (1) and Lemma 2.1, we obtain:

$$\sum_{a\in\mathcal{A}_2^k}\frac{1}{a} > \frac{1}{2}\left(x + \log 2\right)\left(\log\log p_k + \beta - \frac{1}{2\log^2 p_k} + \frac{1}{p_k}\right)\sum_{i=1}^{k}\frac{1}{(x + \log 2)\, p_i},$$

so

$$\sum_{a\in\mathcal{B}_2^k}\frac{1}{a(\log a+x)} \geq \frac{(x+\log 2)}{2(2\log p_k+x)}\sum_{a\in\mathcal{A}_2^k}\frac{1}{a}+\sum_{n>k}\frac{1}{p_n(\log p_n+x)}$$

$$> \frac{(x+\log 2)\left(\log\log p_k+\beta-\frac{1}{2\log^2 p_k}+\frac{1}{p_k}\right)}{2(2\log p_k+x)}\sum_{n=1}^{k}\frac{1}{p_n(\log p_n+x)}+\sum_{n>k}\frac{1}{p_n(\log p_n+x)}.$$

To obtain the inequality required in Theorem 1.1, it is necessary to choose k and x so that

$$\frac{(x+\log 2)\left(\log\log p_k+\beta-\frac{1}{2\log^2 p_k}+\frac{1}{p_k}\right)}{2(2\log p_k+x)} > 1.$$

It is clear that the function

$$x \longmapsto h_k(x) = \frac{\left(\log\log p_k+\beta-\frac{1}{2\log^2 p_k}+\frac{1}{p_k}\right)(x+\log 2)}{2(2\log p_k+x)} \text{ for } k>1,$$

is strictly increasing for $x\in\mathbb{R}^*$. Let choose k and x_0 such that x_0 is an integer and the minimum for which the above inequality holds. Then

$$\frac{\log\log p_k+\beta-\frac{1}{2\log^2 p_k}+\frac{1}{p_k}-2}{4\log p_k-\log 2\left(\log\log p_k+\beta-\frac{1}{2\log^2 p_k}+\frac{1}{p_k}\right)} > \frac{1}{x_0}. \tag{2}$$

Since $x_0>0$, so for the k we choose, we must verify that:

$$\log\log p_k-\frac{1}{2\log^2 p_k}+\frac{1}{p_k}-2+\beta>0.$$

The growth of the function $t\to g(t)=\log\log t-\frac{1}{2\log^2 t}+\frac{1}{t}-2+\beta$ for $t>1$, gives us $p_k\geq 317$ then $k\geq 66$. So for $k\geq 66$ the inequality (2) equivalent to $U_k<x_0$, where

$$U_k=\frac{4\log p_k-\log 4}{\log\log p_k+\beta-\frac{1}{2\log^2 p_k}+\frac{1}{p_k}-2}-\log 2.$$

Let $k_1=373707$, then for $y\geq\log(p_{k_1})$, the function

$$y\longmapsto f(y)=\frac{4y-\log 4}{\log y+\beta-\frac{1}{2y^2}+\frac{1}{e^y}-2}-\log 2,$$

is strictly increasing, because for $y\geq\log(p_{k_1})$, we have $-6+2\beta+2\ln y>0$ and

$$f'(y)=8y^2\frac{e^y\left(\ln 2\frac{e^y}{y}+(-6+2\ln y+2\beta)\,y^2e^y+(2-\ln 2)\,y^2+2y^3+(y\ln 2-3)\,e^y\right)}{\left(2y^2-4y^2e^y-e^y+2y^2e^y\ln y+2y^2\beta e^y\right)^2}>0.$$

So the sequence $(U_k)_{k\geq 373707}$ is strictly increasing, then we get $\min\limits_{k\geq 66}U_k=\min\limits_{66\leq k\leq 373707}U_k$. A computer calculation gives $\min\limits_{k\geq 66}U_k\simeq 59.9$, then $x_0=60$. Let us choose k_0 to be the smallest integer such that $U_k<60$. A computer calculation gives $k_0=130947$ and $p_{k_0}=1740611$. So, if we take $\mathcal{A}=\mathcal{B}_2^{k_0}$, we get $S(\mathcal{B}_2^{k_0},x)>S(\mathcal{P},x)$ for $x\geq x_0$. According to Lemma 2.4, we have:

$$S(\mathcal{B}_2^{k},x)>S(\mathcal{B}_2^{k_0},x)>S(\mathcal{P},x) \text{ for } k\geq k_0 \text{ and } x\geq x_0.$$

This completes the proof. $\square$

Remark 1. *By Lemma 2.4, we can ask the following question. Is it true that for $x > 0$, there exists k such that $S(\mathcal{B}_2^k, x) > S(\mathcal{P}, x)$.*

References

[1] Clark, D. A. (1995). An upper bound of $\sum 1/(a_i \log a_i)$ for primitive sequences, *Proceedings of the American Mathematical Society*, 123, 363–365.

[2] Cohen, H. (1991). *High precision computation of Hardy–Littlewood constants*. Preprint. Available online at: `http://www.math.u-bordeaux1.fr/~cohen/hardylw.dvi`

[3] Erdős, P. (1935). Note on sequences of integers no one of which is divisible by any other. *Journal of the London Mathematical Society*, 10(2), 126–128.

[4] Erdős, P. (1988). Seminar at the University of Limoges.

[5] Erdős, P., & Zhang, Z. (1993). Upper bound of $\sum 1/(a_i \log a_i)$ for primitive sequences. *Proceedings of the American Mathematical Society*, 117(4), 891–895.

[6] Farhi, B. (2017). *Results and conjectures related to a conjecture of Erdős concerning primitive sequences*. Preprint. arXiv:1709.08708v2 [math.NT].

[7] Laib, I., Derbal, A., & Mechik, R. (2019). Somme translatée sur des suites primitives et la conjecture d'Erdős. *Comptes Rendus Mathematique*, 357(5), 413–417.

[8] Lichtman, J. D., & Pomerance, C. (2019). The Erdős conjecture for primitive sets. *Proceedings of the American Mathematical Society Ser. B*, 6, 1–14.

[9] Rezzoug, N., Laib, I., & Guenda, K. (2020). On a translated sum over primitive sequences related to a conjecture of Erdős. *Notes on Number Theory and Discrete Mathematics*, 26(4), 68–73.

[10] Rosser, J. B., & Schoenfeld, L. (1962). Approximates Formulas for Some Functions of Prime Numbers. *Illinois Journal of Mathematics*, 6, 64–94.

[11] Zhang, Z. (1991). On a conjecture of Erdős on the sum $\sum\limits_{p \leq n} 1/(p \log p)$. *Journal of Number Theory*, 39, 14–17.

[12] Zhang, Z. (1993). On a problem of Erdős concerning primitive sequences. *Mathematics of Computation*, 60(202), 827–834.