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Multiplicative functions satisfying the functional equation $\kappa f(m^2+n^2) = f(\kappa m^2) + \kappa f(n^2)$

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Abstract: For a fixed positive integer κ, the functional equation

$$\kappa f(m^2+n^2) = f(\kappa m^2) + \kappa f(n^2) \qquad (m, n \in \mathbb{N})$$

is solved for multiplicative functions f. This complements a 1996 result of Chung [2] which deals with the case $\kappa = 1$. The method used relies on the sum of two squares theorem in number theory.
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1 Introduction

An arithmetic function, [9], is a complex-valued function whose domain is the set of positive integers, $\mathbb{N}$. A multiplicative function is a non-zero arithmetic function which satisfies the multiplicative relation

$$f(mn) = f(m)f(n) \qquad \text{for all } m, n \in \mathbb{N} \text{ with } \gcd(m, n) = 1. \tag{1}$$

A multiplicative function is said to be completely multiplicative if the relation (1) holds for all $m, n \in \mathbb{N}$ without the restriction on their greatest common divisor. Following Spiro [13], a set $E \subseteq \mathbb{N}$ is called an additive uniqueness set for a set S of arithmetic functions if the only function $f \in S$ which satisfies the additive condition

$$f(m+n) = f(n) + f(n) \qquad \text{for any } m, n \in E, \tag{2}$$

is the identity function $f(n) = n$; Spiro proved in the paper that the set of primes is an additive uniqueness set for the set of multiplicative functions non–zero at some prime. In 2011, Fang [4] proved the same result when the functional equation (2) contains three primes. Later, Dubickas and Šarka [3] settled the general case by establishing the same result for the additive condition containing k (≥ 2) primes.

In another direction, in 1996 Chung [2] characterized all multiplicative functions f satisfying

$$f(m^2 + n^2) = f(m^2) + f(n^2) \qquad (m, n \in \mathbb{N}). \tag{3}$$

Chung proved that there are only two possible categories of solutions, the first of which contains the identity function. Slightly different but simliar to Chung's functional equation (3) is the functional equation

$$f(m^2 + n^2) = f(m)^2 + f(n)^2 \qquad (m, n \in \mathbb{N}),$$

which was solved by Bašić, [1] without the restriction on the multiplicative property of the solutions. Bašić found that solution functions fall roughly into three categories: $f \equiv 0$, or $f(n) = \pm n$, or $f(n) = \pm\frac{1}{2}$. A similar functional equation containing a sum of k (≥ 3) squares

$$f(a_1^2 + a_2^2 + \cdots + a_k{}^2) = f(a_1)^2 + f(a_2)^2 + \cdots + f(a_k)^2,$$

was treated by Park [12] who showed that the only multiplicative solution function is the identity function. Along the same vein, Park in [11] proved that if a multiplicative function f satisfies the functional equation, with $k \geq 3$,

$$f(a_1^2 + a_2^2 + \cdots + a_k{}^2) = f(a_1{}^2) + f(a_2{}^2) + \cdots + f(a_k{}^2)$$

which is a k-dimensional version of the equation (3), then it must be the identity function.

For other related recent works, we refer to [5–8].

In the present work, we solve the functional equation

$$\kappa f(m^2 + n^2) = f(\kappa m^2) + \kappa f(n^2) \qquad \text{for all } m, n \in \mathbb{N} \tag{4}$$

for a multiplicative function f, where κ is a positive integer kept fixed throughout. Chung's functional equation (3) corresponds to the case $\kappa = 1$. Our main result is

Theorem 1.1. *Let $\kappa \in \mathbb{N}$ be fixed, and let f be a multiplicative function. Let $\mathcal{F}_1$ be the family of multiplicative functions f such that*

A1) $f(2^t) = 2^t$ *for all integers $t \geq 0$;*

A2) $f(p^t) = p^t$ *for all primes $p \equiv 1 \pmod 4$ and all $t \in \mathbb{N}$;*

A3) $f(q^{2t}) = q^{2t}$ *for all primes $q \equiv 3 \pmod 4$ and all $t \in \mathbb{N}$,* *let $\mathcal{F}_2$ be the family of multiplicative functions f such that*

B1) $f(2) = 2,\ f(2^t) = 0$ *for all integers* $t \geq 2$*;*

B2) $f(p^t) = 1$ *for all primes* $p \equiv 1 \pmod 4$ *and all* $t \in \mathbb{N}$*;*

B3) $f(q^{2t}) = 1$ *for all primes* $q \equiv 3 \pmod 4$ *and all* $t \in \mathbb{N}$*,*

and let $\mathcal{F}_3$ be the family of multiplicative functions f such that

C1) $f(2) = 1 + \frac{1}{\kappa} f(\kappa),\ f(2^t) = 2 - f(2)$ *for all integers* $t \geq 2$*;*

C2) $f(p^t) = 1$ *for all primes* $p \equiv 1 \pmod 4$ *and all* $t \in \mathbb{N}$*;*

C3) $f(q^{2t}) = 1$ *for all primes* $q \equiv 3 \pmod 4$ *and all* $t \in \mathbb{N}$*.*

Then f satisfies the functional equation (4) *if and only if f fulfills the conditions*

$$f(a^{2\alpha}\kappa) = \kappa f(a^{2\alpha}) + f(\kappa) - \kappa \qquad \text{for all } a, \alpha \in \mathbb{N}, \tag{5}$$

and belongs to one of the following families:

Fam1) $f \in \mathcal{F}_1$ *and* $f(\kappa) = \kappa$*, or*

Fam2) $f \in \mathcal{F}_2$ *and* $f(\kappa) = \kappa$ *(provided that* $\kappa = 1$ *or* 2*), or*

Fam3) $f \in \mathcal{F}_3$ *and* $f(\kappa) \neq \kappa$*.*

2 Auxiliary lemmas

In this section, we gather some auxiliary results needed in the proof of our main theorem; the first is taken from [2].

Lemma 2.1 ([2, Theorem, Corollary])**.** *I. Let f be a multiplicative function. Then f satisfies the functional equation*

$$f(m^2 + n^2) = f(m^2) + f(n^2) \qquad \text{for all } m, n \in \mathbb{N} \tag{6}$$

if and only if either $f \in \mathcal{F}_1$, or $f \in \mathcal{F}_2$.

II. Let f be a completely multiplicative function. Then f satisfies (6) *if and only if* $f(2) = 2$, $f(p) = p$ *for all primes* $p \equiv 1 \pmod 4$*, and* $f(q) \in \{q, -q\}$ *for all primes* $q \equiv 3 \pmod 4$.

The next lemma is known as the sum of two squares theorem, due to Fermat, which will be prominent in our present work.

Lemma 2.2 ([10, Theorem 2.15])**.** *Write the canonical prime factorization of n in the form*

$$n = 2^{\alpha} \prod_{p \equiv 1 \bmod 4} p^{\beta} \prod_{q \equiv 3 \bmod 4} q^{\gamma},$$

where the product runs through distinct primes p's and q's. Then n can be expressed as a sum of two squares of integers if and only if all the exponents γ are even.

The next lemma is used often in the proof of Lemma 2.4 and we omit its easy proof.

Lemma 2.3. *For an odd integer N, we can write*

$$N^2+1=2\left(\left(\frac{N+1}{2}\right)^2+\left(\frac{N-1}{2}\right)^2\right), \tag{7}$$

where $(N\pm 1)/2$ are integers and $\gcd\left(((N+1)/2)^2+((N-1)/2)^2, 2\right)=1$.

For our main functional equation (4), the following properties are basic.

Lemma 2.4. *Suppose f satisfies* (4). *Let $A_\kappa := \frac{1}{\kappa}(f(\kappa)-\kappa)$. Then for all $a,\alpha\in\mathbb{N}$ we have*

P1) $f(a^{2\alpha}\kappa)=\kappa f(a^{2\alpha})+f(\kappa)-\kappa$;

P2) $f(a^{2\alpha})=f(a^{2\alpha-2})(f(a^2+1)-1)-A_\kappa$;

P3) $A_\kappa=f(2)-2,\ f(\kappa)=\kappa(f(2)-1)$;

P4) $f(2^{2\alpha})=f(2^{2\alpha-2})(f(4)+A_\kappa)-A_\kappa$;

P5) *i)* $f(5)=f(4)+f(2)-1$,

ii) $f(9)=f(4)f(2)+f(2)^2-2f(2)+1$,

iii) $f(13)=f(4)(f(2)+1)+f(2)(A_\kappa+1)-1$,

iv) $f(25)=f(2)[f(4)(f(2)+1)+f(2)(A_\kappa+1)-1]-A_\kappa-1$;

P6) $f(2)\in\{2,2-f(4)\}$.

Proof. To prove P1), using (4) twice, and using the fact that $f(1)=1$ for multiplicative functions, we have $f(\kappa\cdot 1^2)+\kappa f(a^{2\alpha})=\kappa f(a^{2\alpha}+1^2)=f(\kappa a^{2\alpha})+\kappa f(1^2)$, and the assertion follows.

To prove P2), from P1), using (4), and the multiplicativity of f (since $\gcd(a^{2\alpha-2},a^2+1)=1$), we get

$$\begin{aligned}\kappa f(a^{2\alpha})+f(\kappa)-\kappa+\kappa f(a^{2\alpha-2})&=f(a^{2\alpha}\kappa)+\kappa f(a^{2\alpha-2})=\kappa f(a^{2\alpha}+a^{2\alpha-2})\\&=\kappa f(a^{2\alpha-2})f(a^2+1),\end{aligned}$$

and the assertion follows after simplification.

To prove P3), note that putting $m=n=1$ into (4), we get $f(2)=\frac{1}{\kappa}f(\kappa)+1$, and so the definition of A_κ gives $A_\kappa=\frac{1}{\kappa}f(\kappa)-1=f(2)-2$.

To prove P4), first note that putting $m=2,\ n=1$ in (4) and using P1), we get

$$\kappa f(5)=\kappa f(2^2+1^2)=f(4\kappa)+\kappa f(1)=\kappa f(4)+f(\kappa)-\kappa+\kappa,$$

which gives $f(5)=f(4)+1+A_\kappa=f(4)+f(2)-1$. Substituting this into P2) with $a=2$ yields the assertion.

Regarding P5), assertion *i)* has just been verified in the proof of P4). To prove assertion *ii)*, we start by using *i)*, multiplicativity, (4), and P1) to get

$$\begin{aligned}\kappa f(2)(f(4)+1+A_\kappa)=\kappa f(2)f(5)=\kappa f(10)=\kappa f(3^2+1^2)&=f(3^2\kappa)+\kappa f(1^2)\\&=\kappa f(9)+f(\kappa).\end{aligned}$$

Simplifying using P3), the assertion follows. For assertion *iii)*, using (4) and P1), we have

$$\kappa f(13) = \kappa f(3^2 + 2^2) = f(3^2\kappa) + \kappa f(2^2) = \kappa f(9) + f(\kappa) - \kappa + \kappa f(4),$$

and the result follows by simplifying using *ii)* and the definition of A_κ. For assertion *iv)*, we start with *iii)*, multiplictivity, (4) and P1), to get

$$\begin{aligned}&\kappa f(2)\left\{f(4)(f(2)+1) + f(2)(A_\kappa + 1) - 1\right\} = \kappa f(2)f(13) = \kappa f(26)\\ &= \kappa f(5^2 + 1^2) = f(5^2\kappa) + \kappa f(1^2) = \kappa f(5^2) + f(\kappa),\end{aligned}$$

and the assertion follows by simplifying using the definition of A_κ.

To prove P6), using multiplicativity, the functional equation and P1), we get

$$\kappa f(2)f(9) = \kappa f(18) = \kappa f(3^2 + 3^2) = f(3^2\kappa) + \kappa f(3^2) = 2\kappa f(3^2) + f(\kappa) - \kappa.$$

Replacing A_κ in P5) part *ii)* and using P3), we have

$$f(9) = f(4)f(2) + f(2)^2 - 2f(2) + 1.$$

Using P3) and simplifying, we get

$$0 = \kappa f(2)f(9) - 2\kappa f(3^2) - f(\kappa) + \kappa = \kappa f(2)(f(2)-2)(f(4) + f(2) - 2),$$

and so

$$f(2) = 0, \quad \text{or} \quad f(2) = 2 \quad \text{or} \quad f(2) = 2 - f(4). \tag{8}$$

On the other hand, from the functional equation (4), P1) and P3), we have

$$\kappa f(8) = \kappa f(2^2 + 2^2) = f(2^2\kappa) + \kappa f(2^2) = 2\kappa f(4) + \kappa f(2) - 2\kappa, \tag{9}$$

and

$$\kappa f(5)f(8) = \kappa f(40) = \kappa f(2^2 + 6^2) = f(2^2\kappa) + \kappa f(6^2) = \kappa f(4) + f(\kappa) - \kappa + \kappa f(4)f(9).$$

In this last relation, replacing $\kappa f(8)$ from (9), and simplifying, we get

$$f(5)\left(2f(4) + f(2) - 2\right) = f(4) + f(2) - 2 + f(4)f(9).$$

Substituting the values of $f(5)$ and $f(9)$ from P5), and simplifying, we arrive at

$$(f(2)-2)(f(4)-1)(f(4)+f(2)-2) = 0. \tag{10}$$

If $f(2) \neq 2$ and $f(2) \neq 2 - f(4)$, then from (8) we must have $f(2) = 0$ and from (10) we must have $f(4) = 1$. The assertions P5) *ii)* and *iv)* now yield $f(9) = 1 = f(25)$, and so, using also P3), the expressions P1) and P4) respectively become

$$f(a^{2\alpha}\kappa) = \kappa f(a^{2\alpha}) - 2\kappa, \quad f(2^{2\alpha}) = f(2^{2\alpha-2})(f(4)-2) + 2. \tag{11}$$

Applying (11) to the first term on the right-hand side of (4) and simplifying, we get

$$f(m^2+n^2) = f(m^2) + f(n^2) - 2. \tag{12}$$

Replacing $m = 4, n = 3$, we obtain $f(16) = f(25) - f(9) + 2 = 2$. In another direction, by (11), we have $f(16) = f(2^4) = f(2^2)(f(4) - 2) + 2$. Equating the last two equalities yields $f(4) \in \{0, 2\}$. This give a contradiction to $f(4) = 1$, and thus we conclude that

$$f(2) = 2 \quad \text{or} \quad f(2) = 2 - f(4). \qquad \square$$

From the last lemma, there are two possible values of $f(2)$, the next lemma deals with each of these possibilities.

Lemma 2.5. *Assume that the multiplicative function f satisfies the functional equation* (4).

I) If $f(\kappa) = \kappa$, then either $f \in \mathcal{F}_1$, or $f \in \mathcal{F}_2$ (provided that $\kappa = 1$ or 2).

II) If $f(\kappa) \neq \kappa$, then $f \in \mathcal{F}_3$.

Proof. I) Since $f(\kappa) = \kappa$, the property P1) in Lemma 2.4 reduces to $f(a^{2\alpha}\kappa) = \kappa f(a^{2\alpha})$. Simplifying (4) using this last relation leads to $f(m^2 + n^2) = f(m^2) + f(n^2)$. The desired result follows by appealing to Lemma 2.1.

II) Since $f(\kappa) \neq \kappa$, the property P3) in Lemma 2.4 implies $f(2) \neq 2$, and so the property P6) yields $f(2) = 2 - f(4)$ with $f(4) \neq 0$. We now claim that

II-0) $f(2) = 1 + \frac{1}{\kappa} f(\kappa)$;

II-i) $f(2^t) = 2 - f(2)$ for all integers $t \geq 2$;

II-ii) $f(m^2) = 1$ for all odd $m \in \mathbb{N}$;

II-iii) $f(n^2) = 2 - f(2)$ for all even $n \in \mathbb{N}$;

II-iv) $f(p^t) = 1$ for all primes $p \equiv 1 \pmod 4$ and all $t \in \mathbb{N}$;

II-v) $f(q^{2t}) = 1$ for all primes $q \equiv 3 \pmod 4$ and all $t \in \mathbb{N}$.

To prove II-0), putting $m = n = 1$ into (4), the result follows immediately.

To prove II-i), note that the properties P3) and P4) give

$$f(2^{2\alpha}) = -A_\kappa = 2 - f(2) \tag{13}$$

showing that II-i) holds for all even $t \in \mathbb{N}$. We now argue by induction on t. Assume that $f(2^t) = 2 - f(2)$ holds for all $t = 2, 3, \ldots, n$. We show that $f(2^{n+1}) = 2 - f(2)$. This is immediate if $n + 1$ is even. Assume then that $n + 1 = 2s + 1$ is odd. The equation (4), P1), P3) and the induction hypothesis together give

$$\kappa f(2^{n+1}) = \kappa f(2^{2s} + 2^{2s}) = f(2^{2s}\kappa) + \kappa f(2^{2s}) = 2\kappa f(2^{2s}) + \kappa A_\kappa = -\kappa A_\kappa,$$

proving II-i).

To prove II-ii), note that by multiplicativity, it trivially holds when $m = 1$. Assume that $f(m^2) = 1$ for all odd $m \in \{1, 3, \dots, n-2\},\ n$ odd ≥ 3. We show that $f(n^2) = 1$. Since n is odd, from P1), using (4), Lemma 2.3, the multiplicativity of f, (4) and P1) again, we have

$$\begin{aligned}
\kappa f(n^2) + f(\kappa) &= f(\kappa n^2) + \kappa = \kappa f(n^2+1) \\
&= \kappa f(2) f\left(\left(\frac{n+1}{2}\right)^2 + \left(\frac{n-1}{2}\right)^2\right) \\
&= f(2)\left(f\left(\kappa\left(\frac{n+1}{2}\right)^2\right) + \kappa f\left(\left(\frac{n-1}{2}\right)^2\right)\right) \\
&= f(2)\left(\kappa f\left(\left(\frac{n+1}{2}\right)^2\right) + \kappa f\left(\left(\frac{n-1}{2}\right)^2\right) + f(\kappa) - \kappa\right). \qquad (14)
\end{aligned}$$

If $n \equiv 1 \pmod 4$, then $(n-1)/2 \in \mathbb{N}$ is even, while $(n+1)/2 \in \mathbb{N}$ is odd. Substituting $(n-1)/2 = 2^\ell s$, where s is odd $< (n-1)/2 < n$ and $\ell \in \mathbb{N}$, the right-hand side of (14) becomes

$$f(2)\left(\kappa f\left(((n+1)/2)^2\right) + \kappa f(2^{2\ell}) f(s^2) + f(\kappa) - \kappa\right)$$

Applying the induction hypothesis, II-i), P3), and the definition of A_κ to this last expression, we get

$$\kappa f(n^2) + f(\kappa) = f(2)\left(\kappa + \kappa(2 - f(2)) + f(\kappa) - \kappa\right) = \kappa f(2).$$

Using $f(\kappa) = \kappa(f(2) - 1)$, which results from P3) and the definition of A_κ, the assertion II-ii) follows in this case. The case where $n \equiv 3 \pmod 4$ is proved analogously.

To prove II-iii), for even $n \in \mathbb{N}$, write $n = 2^\ell s$, where $s < n$ is odd and $\ell \in \mathbb{N}$. Using multiplicativity, II-i) and II-ii), we have $f(n^2) = f(2^{2\ell}) f(s^2) = 2 - f(2)$.

To prove II-iv), we make use of Lemma 2.2. For any prime $p \equiv 1 \pmod 4$, there exist odd $m_o \in \mathbb{N}$ and even $n_e \in \mathbb{N}$ such that $p^t = m_o^2 + n_e^2$ for all $t \in \mathbb{N}$. By (4), Lemma 2.4 P1), P3), and simplifying, we have

$$\kappa f(p^t) = \kappa f(m_o^2 + n_e^2) = f(\kappa m_o^2) + \kappa f(n_e^2) = \kappa f(m_o^2) + \kappa f(n_e^2) + f(\kappa) - \kappa$$

which is equivalent to

$$f(p^t) = f(m_o^2) + f(n_e^2) + A_\kappa = f(m_o^2) + f(n_e^2) + f(2) - 2.$$

Using II-ii) and II-iii), we arrive at

$$f(p^t) = 1 + 2 - f(2) + f(2) - 2 = 1.$$

The assertion II-v) follows immediately from II-ii).

This completes the proof. □

3 Proofs of Theorem 1.1 and its corollary

Assume that the multiplicative function f satisfies the functional equation (4). Lemma 2.4 P1) shows that (5) holds. If $f(\kappa)=\kappa$, then Lemma 2.5 I shows either $f\in\mathcal{F}_1$, or $\mathcal{F}_2$ (provided that $\kappa=1$ or 2). If $f(\kappa)\neq\kappa$, then Lemma 2.5 II ensures that $f\in\mathcal{F}_3$.

Conversely, assume that the multiplicative function f satisfies (5) and belongs to one of the families Fam1), Fam2) or Fam3).

If f belongs to Fam1), i.e., $f\in\mathcal{F}_1$ and $f(\kappa)=\kappa$. The functional values in A1), A2) and A3) imply immediately that $f(b^2)=b^2\quad(b\in\mathbb{N})$. Incorporating into (5), we get

$$f(\kappa m^2)=\kappa f(m^2)+f(\kappa)-\kappa=\kappa m^2.$$

By Lemma 2.2 we can write

$$m^2+n^2=2^t\prod_i p_i^{\alpha_i}\prod_j q_j^{2\beta_j}, \tag{15}$$

where $p_i\equiv 1 \bmod 4,\ q_j\equiv 3 \bmod 4$ are distinct primes and $\alpha_i,\beta_j,t\in\mathbb{N}$. Using the multiplicativity of f, A1), A2) and A3), we have

$$\begin{aligned}\kappa f(m^2+n^2)&=\kappa f(2^t)\prod_i f(p_i^{\alpha_i})\prod_j f(q_j^{2\beta_j})=\kappa 2^t\prod_i p_i^{\alpha_i}\prod_j q_j^{2\beta_j}=\kappa m^2+\kappa n^2\\&=f(\kappa m^2)+\kappa f(n^2),\end{aligned}$$

showing that f satisfies (4).

The proof of the case where f belongs to the family Fam2) is simpler but similar to that of f belonging to Fam3) so we show only that of the latter case.

If f belongs to Fam3), the functional values in C1), C2) and C3) imply that $f(c^2)=1$ for any odd $c\in\mathbb{N}$. Using the sum of two squares representation (15), C1), C2), C3) and the multiplicativity of f, we have

$$\kappa f(m^2+n^2)=\kappa f(2^t).$$

If $t=0$, then one of m and n must be odd and the other even; if m is odd and n is even, write $n=2^a b$, where b is odd and $a\in\mathbb{N}$. By (5), C1), C2), C3), and the multiplicity of f, we have

$$\begin{aligned}f(\kappa m^2)+\kappa f(n^2)&=\kappa f(m^2)+f(\kappa)-\kappa+\kappa f(2^{2a})f(b^2)=f(\kappa)+\kappa(2-f(2))=\kappa\\&=\kappa f(m^2+n^2),\end{aligned}$$

showing that f satisfies (4). The possibility of even m and odd n is omitted as it is almost the same.

If $t=1$, then both m and n must be odd. By (5), C1), C2) and C3), we have

$$\begin{aligned}f(\kappa m^2)+\kappa f(n^2)&=\kappa f(m^2)+f(\kappa)-\kappa+\kappa=\kappa+f(\kappa)=\kappa f(2)\\&=\kappa f(m^2+n^2),\end{aligned}$$

showing that f satisfies (4).

If $t \geq 2$, then both m and n must be even, say, $m = 2^u v,\ n = 2^x y$ with odd v, y and $u, x \in \mathbb{N}$. By (5), C1), C2), C3), and the multiplicity of f, we have

$$\begin{aligned} f(\kappa m^2) + \kappa f(n^2) &= \kappa f(m^2) + f(\kappa) - \kappa + \kappa f(n^2) = \kappa f(2^{2u}) f(v^2) + f(\kappa) - \kappa + \kappa f(2^{2x}) f(y^2) \\ &= \kappa(2 - f(2)) + f(\kappa) - \kappa + \kappa(2 - f(2)) = \kappa(2 - f(2)) = \kappa f(2^t) \\ &= \kappa f(m^2 + n^2), \end{aligned}$$

showing that f satisfies (4).

The work of Chung [2] corresponds to the cases Fam1) and Fam2) of our Theorem 1.1. If $\kappa = 1, 2$, then $\mathcal{F}_3$ is identical with $\mathcal{F}_2$. If $\kappa \in \mathbb{N}\backslash\{1, 2\}$, then $\mathcal{F}_3$ is different from both $\mathcal{F}_1$ and $\mathcal{F}_2$.

For completely multiplicative functions, we have:

Corollary 3.1. *Assume that f is a completely multiplicative function. Then f satisfies the functional equation* (4) *if and only if f fulfills the condition* (5) *and the following assertions hold:*

(i) $f(2) = 2$*;*

(ii) $f(p) = p$ *for all primes* $p \equiv 1 \pmod 4$*;*

(iii) $f(q) \in \{-q, q\}$ *for all primes* $q \equiv 3 \pmod 4$.

Proof. If f satisfies (4), the condition (5) follows immediately from Lemma 2.4 P1). Being completely multiplicative, the equation (4) reduces to $f(m^2 + n^2) = f(m^2) + f(n^2)$. Assertions *(i)*, *(ii)*, and *(iii)* follow at once from Lemma 2.1 II.

Conversely, assume that the completely multiplicative function f satisfies (5), *(i)*, *(ii)*, and *(iii)*. Observe that (5) yields $f(\kappa) = \kappa$, while *(i)*, *(ii)*, and *(iii)* yield $f(b^2) = b^2$ $(b \in \mathbb{N})$. Using Lemma 2.2, write $m^2 + n^2 = 2^t p_1^{\alpha_1} \cdots p_\ell^{\alpha_\ell} q_1^{2\beta_1} \cdots q_s^{2\beta_s}$, where p_i, q_j are distinct primes with $p_i \equiv 1 \bmod 4,\ q_j \equiv 3 \bmod 4$. Thus,

$$\begin{aligned} \kappa f(m^2 + n^2) &= \kappa f(2)^t \prod_i f(p_i)^{\alpha_i} \prod_j f(q_j)^{2\beta_j} = \kappa 2^t \prod_i p_i^{\alpha_i} \prod_j q_j^{2\beta_j} \\ &= \kappa m^2 + \kappa n^2 = f(\kappa) f(m^2) + \kappa f(n^2) \\ &= f(\kappa m^2) + \kappa f(n^2), \end{aligned}$$

i.e., f satisfies the function equation (4). □

4 Final remarks

The following remarks are given to us by one of the referees, to whom we are grateful. Note that the functional equation

$$\kappa f(n^2 + m^2) = f(\kappa n^2) + \kappa f(m^2)$$

implies

$$f(\kappa m^2) + \kappa f(n^2) = f(\kappa n^2) + \kappa f(m^2),$$

which gives

$$f(\kappa n^2) - \kappa f(n^2) = f(\kappa m^2) - \kappa f(m^2)$$

for all $m, n \in \mathbb{N}$. Putting $m = 1$, we have

$$f(\kappa n^2) = \kappa f(n^2) + f(\kappa) - \kappa,$$

and so

$$\kappa f(n^2 + m^2) = \kappa f(n^2) + \kappa f(m^2) + f(\kappa) - \kappa,$$

which is equivalent to

$$f(n^2 + m^2) = f(n^2) + f(m^2) + A_\kappa, \tag{16}$$

where $A_\kappa := \frac{1}{\kappa}(f(\kappa) - \kappa)$.

If $A_\kappa = 0$, i.e., $f(\kappa) = \kappa$, then the equation (16) becomes to the equation (3) treated by Chung [2], and the solutions are given by the families $\mathcal{F}_1, \mathcal{F}_2$.

If $A_\kappa \neq 0$, i.e., $f(\kappa) \neq \kappa$, then the solutions belong to the family $\mathcal{F}_3$ obtained from Lemma 2.5.

Since (16) is a special case of the functional equation

$$F(n^2 + m^2 + k) = F(n^2) + F(m^2) + K \qquad (k \in \mathbb{N} \cup \{0\},\ K \in \mathbb{C}), \tag{17}$$

finding solutions of this last equation is still an open problem worthy of future research. Similar to (17) is the following functional equation

$$F(n^2 + m^2 + k) = F(n)^2 + F(m)^2 + K \qquad (k \in \mathbb{N} \cup \{0\},\ K \in \mathbb{C}), \tag{18}$$

which was solved for an arithmetic function $F : \mathbb{N} \to \mathbb{C}$ by Khanh [7] in 2019.

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References

[1] Bašić, B. (2014). Characterization of arithmetic functions that preserve the sum-of-squares operation. *Acta Mathematica Sinica*, 4, 689–695.

[2] Chung, P. V. (1996). Multiplicative functions satisfying the equation $f(m^2+n^2) = f(m^2)+f(n^2)$. *Mathematica Slovaca*, 46, 165–171.

[3] Dubickas, A., & Šarka, P. (2013). On multiplicative functions which are additive on sums of primes. *Aequationes Mathematicae*, 86, 81–89.

[4] Fang, J. H. (2011). A characterization of the identity function with equation $f(p+q+r) = f(p) + f(q) + f(r)$. *Combinatorica*, 31, 697–701.

[5] Kátai, I.,& Phong, B. M. (2014). The functional equation $f(A + B) = g(A) + h(B)$. *Annales Universitatis Scientiarium Budapestinensis de Rolando Eötvös nominatae. Sectio Computatorica*, 43, 287–301.

[6] Khanh, B. M. M. (2015). On the equation $f(n^2 + Dm^2) = f(n)^2 + Df(m)^2$. *Annales Universitatis Scientiarium Budapestinensis de Rolando Eötvös nominatae. Sectio Computatorica*, 44, 59–68.

[7] Khanh, B. M. M. (2019). A note on a result of B. Bojan. *Annales Universitatis Scientiarium Budapestinensis de Rolando Eötvös nominatae. Sectio Computatorica*, 49, 285–297.

[8] Kim, B., Kim, J. Y., Lee, C. G., & Park, P.-S. (2018). Multiplicative functions additive on generalized pentagonal numbers. *Comptes Rendus Mathematique*, 356(2), 125–128.

[9] McCarthy P. J. (1985). *Introduction to Arithmetical Functions*. Springer, New York.

[10] Niven, I., Zuckerman, H. S., & Montgomery, H. L. (1991). *An Introduction to the Theory of Numbers*. Wiley, New York.

[11] Park, P.-S. (2018). Multiplicative functions commutable with sums of squares. *International Journal of Number Theory*, 2, 469–478.

[12] Park, P.-S. (2018). On k-additive uniqueness of the set of squares for multiplicative functions. *Aequationes Mathematicae*, 92, 487–495.

[13] Spiro, C. (1992). Additive uniqueness sets for arithmetic functions. *Journal of Number Theory*, 42, 232–246.