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# On $r$-dynamic coloring of comb graphs

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**Abstract:** An $r$-dynamic coloring of a graph $G$ is a proper coloring of $G$ such that every vertex in $V(G)$ has neighbors in at least $\min\{d(v), r\}$ different color classes. The $r$-dynamic chromatic number of graph $G$ denoted as $\chi_r(G)$, is the least $k$ such that $G$ has a coloring. In this paper we obtain the $r$-dynamic chromatic number of the central graph, middle graph, total graph, line graph, para-line graph and sub-division graph of the comb graph $P_n \odot K_1$ denoted by $C(P_n \odot K_1)$, $M(P_n \odot K_1)$, $T(P_n \odot K_1)$, $L(P_n \odot K_1)$, $P(P_n \odot K_1)$ and $S(P_n \odot K_1)$ respectively by finding the upper bound and lower bound for the $r$-dynamic chromatic number of the Comb graph.
**Keywords:** $r$-dynamic coloring, Comb graph, Central graph, Middle graph, Total graph, Line graph, Sub-division graph, Para-line graph.
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## 1 Introduction

In this paper, all graphs are simple and finite. For a graph $G$, let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degree of $G$. The $r$-dynamic coloring was first introduced by Montgomery [10]. An $r$-dynamic coloring of a graph is a map $c$ from $V(G)$ to the set of colors such that:

(i) if $uv \in E(G)$, then $c(u) \neq c(v)$, and

(ii) for each vertex $v \in V(G)$, $|c(N(v))| \geq \min\{d(v), r\}$,

where $N(v)$ denotes the set of all vertices adjacent to $v$ and $d(v)$ its degree and $r$ is a positive integer. The first condition characterizes proper coloring and it is called the adjacency condition and second condition is the $r$-adjacency condition. The $r$-dynamic chromatic number of a graph $G$ is denoted by $\chi_r(G)$, is the minimum $k$ such that $G$ admits such a proper $k$-coloring. The 1-dynamic chromatic number of a graph $G$ is equal to its chromatic number. The 2-dynamic chromatic number of a graph $G$ is studied by the name dynamic chromatic number in [1–4, 7].

There are many upper bounds and lower bounds for $\chi_d(G)$ in terms of graph parameters. For a graph $G$ with $\Delta(G) \geq 3$, Lai et al. [7] proved that $\chi_d(G) \leq \Delta(G) + 1$, except for a cycle graph $C_5$. An upper bound for the dynamic chromatic number of a regular graph $G$ and the independence number of the graph $G$, $\alpha(G)$, was introduced in [5]. In fact, it was proved that $\chi_2(G) \leq \chi(G) + 2\log_2 \alpha(G) + \mathcal{O}(1)$. Taherkhani gave [11] an upper bound for $\chi_2(G)$ in terms of the chromatic number, the maximum degree $\Delta$ and the minimum degree $\delta$ that is $\chi_2(G) - \chi(G) \leq [\Delta e/\delta \log(2e(\Delta^2 + 1))]$, where $G$ is again a $d$-regular graph. Li et al. proved in [8] that in determining the value of $\chi_r(G)$ for planar bipartite graphs with maximum degree at most 3 and arbitrary high girth is an NP-hard problem. Furthermore, Li and Zhou [8] showed that to determine whether there exist a 3-dynamic coloring or not, for a claw free graph with the maximum degree 3 is an NP-complete problem.

## 2 Preliminaries

Let $G$ be a simple and finite graph with vertex $V(G)$ and edge set $E(G)$. The middle graph [9] of $G$ denoted by $M(G)$, is defined as follows, the vertex set of $M(G)$ is $V(G) \cup E(G)$. Two vertices $x, y$ of $M(G)$ are adjacent in $M(G)$ in case one of the following holds: (i) $x, y$ are in $E(G)$ and $x, y$ are adjacent in $G$. (ii) $x$ is in $V(G)$, $y$ is in $E(G)$, and $x, y$ are incident in $G$.

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The total graph [9] of $G$, denoted by $T(G)$, is defined in the following way. The vertex set of $T(G)$ is $V(G) \cup E(G)$. Two vertices $x, y$ of $T(G)$ are adjacent in $T(G)$ in case one of the following holds: (i) $x, y$ are in $V(G)$ and $x$ is adjacent to $y$ in $G$. (ii) $x, y$ are in $E(G)$ and $x, y$ are adjacent in $G$. (iii) $x$ is in $V(G)$, $y$ is in $E(G)$, and $x, y$ are incident in $G$.

The central graph [12] $C(G)$ of a graph $G$ is obtained from $G$ by adding an extra vertex on each edge of $G$, and then joining each pair of vertices of the original graph which were previously non-adjacent.

The line graph [6] of $G$, denoted by $L(G)$, is the graph whose vertex set is the edge set of $G$. Two vertices of $L(G)$ are adjacent whenever the corresponding edges of $G$ are adjacent.

The sub-division graph $S(G)$ is obtained simply by inserting a new vertex for each edge of $G$.

The line graph of a sub-division graph is the para-line graph $P(G)$.

Let $P_n$ be a path graph with $n$ vertices and $K_1$ be a complete graph with one vertex. The comb graph $P_n \odot K_1$ is defined as the corona product of path graph $P_n$ with the complete graph $K_1$ by taking one copy of $P_n$ and $|V(P_n)|$ copies of $K_1$ and making the $i^{th}$ vertex of $P_n$ adjacent to the $i^{th}$ copy of $K_1$ where $1 \leq i \leq n$. Comb graph has $2n$ vertices and $2n - 1$ edges.

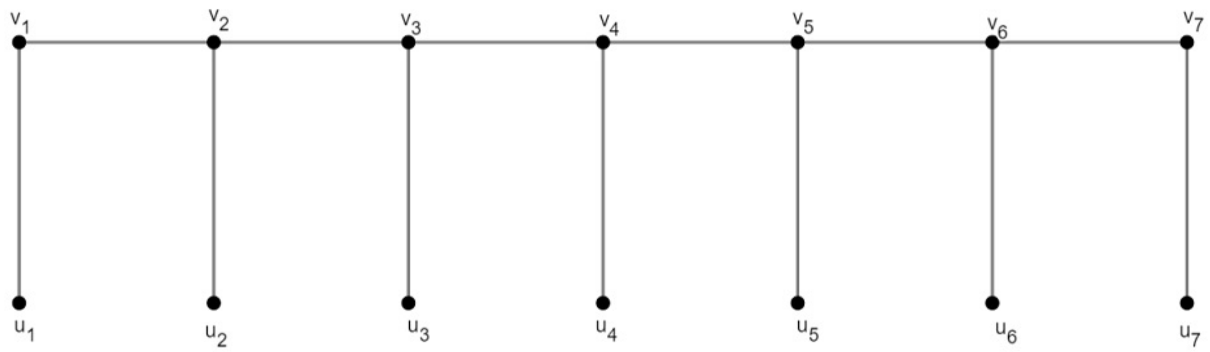


Figure 1. Comb graph $P_7 \odot K_1$

In the following section we obtain the $r$-dynamic chromatic number of the central graph, middle graph, total graph, line graph, para-line graph and sub-division graph of the comb graph $P_n \odot K_1$ denoted by $C(P_n \odot K_1)$, $M(P_n \odot K_1)$, $T(P_n \odot K_1)$, $L(P_n \odot K_1)$, $P(P_n \odot K_1)$ and $S(P_n \odot K_1)$, respectively.

# 3 Main results

**Lemma 3.1.** $\chi_r(G) \geq \min\{r, \Delta(G)\} + 1$.

**Theorem 3.2.** *Let $n \geq 3$, $C(P_n \odot K_1)$ be the central graph of comb graph then*

$$\chi_r(C(P_n \odot K_1)) = \begin{cases} n, & \text{for } r = 1 \\ 2n, & \text{for } 2 \leq r \leq \Delta - 1 \\ 2n+3, & \text{for } r = \Delta \end{cases}$$

*Proof.* Let

$$V(C(P_n \odot K_1)) = \{v_i : 1 \leq i \leq n\} \cup \{v_i^{'} : 1 \leq i \leq n-1\} \cup \{u_i : 1 \leq i \leq n\} \cup \{u_i^{'} : 1 \leq i \leq n\}$$

where $v_i^{'}$ and $u_i^{'}$ are the vertices corresponding to the edge $v_i v_{i+1}$ and $v_i u_i$ of $(P_n \odot K_1)$, $1 \leq i \leq n$. The maximum and minimum degrees of $C(P_n \odot K_1)$ are $\Delta(C(P_n \odot K_1)) = 2n - 1$ and $\delta(C(P_n \odot K_1)) = 2$. Define the mapping $c : V \to Z^+$.

**Case 1:** When $r = 1$, the $r$-dynamic coloring is $n$ are as follows:

- $c(v_i) = i, 1 \leq i \leq n$
- $c(u_i) = i, 1 \leq i \leq n$
- $c(v_i^{'}) = \begin{cases} 4, & \text{for } i \neq 3, 4,\ 1 \leq i \leq n-1 \\ 2, & \text{for } i = 3, 4,\ 1 \leq i \leq n-1 \end{cases}$
- $c(u_i^{'}) = \begin{cases} 3, & \text{for } i \neq 3,\ 1 \leq i \leq n \\ 4, & \text{for } i = 3,\ 1 \leq i \leq n \end{cases}$

Hence the $r$-adjacency condition is fulfilled, therefore $\chi_r(C(P_n \odot K_1)) = n$ for $r = 1$.

If $\chi_r(C(P_n \odot K_1)) < n$, then the $r$-adjacency condition will not be fulfilled.

**Case 2:** When $2 \le r \le \Delta - 1$, the $r$-dynamic coloring is $2n$ are as follows:

- $c(v_i) = i, 1 \le i \le n$
- $c(u_i) = i + n, 1 \le i \le n$
- $c(v_i^{'}) = i + n, 1 \le i \le n$
- $c(u_i^{'}) = \begin{cases} n, & \text{for } i = 1,\ 1 \le i \le n-1 \\ i-1, & \text{for } 2 \le i \le n-1 \end{cases}$

Hence the $r$-adjacency condition is fulfilled, therefore $\chi_r(C(P_n \odot K_1)) = 2n$ for $2 \le r \le \Delta - 1$. If $\chi_r(C(P_n \odot K_1)) < 2n$, then the $r$-adjacency condition will not be fulfilled.

**Case: 3** When $r = \Delta$, the $r$-dynamic coloring is $2n + 3$ are as follows:

- $c(v_i) = i, 1 \le i \le n$
- $c(u_i) = i + n, 1 \le i \le n$
- $c(u_i^{'}) = 2n + 1, 1 \le i \le n$
- $c(v_i^{'}) = \begin{cases} 2n+2, & \text{for } i \text{ odd} \\ 2n+3, & \text{for } i \text{ even} \end{cases}$

Hence the $r$-adjacency condition is fulfilled, therefore $\chi_r(C(P_n \odot K_1)) = 2n+3$ for $r = \Delta$. If $\chi_r(C(P_n \odot K_1)) < 2n + 3$, then the $r$-adjacency condition will not be fulfilled. □

**Lemma 3.3.** *Let $n \ge 4$, $M(P_n \odot K_1)$ be the middle graph of comb graph then*

$$\chi_r(M(P_n \odot K_1)) \ge \begin{cases} 4, & \text{for } 1 \le r \le 3 \\ r+1, & \text{for } 4 \le r \le \Delta \end{cases}$$

*Proof.* Let

$$V(M(P_n \odot K_1)) = \{v_i : 1 \le i \le n\} \cup \{v_i^{'} : 1 \le i \le n-1\} \cup \{u_i : 1 \le i \le n\} \cup \{u_i^{'} : 1 \le i \le n\}$$

where $v_i^{'}$ and $u_i^{'}$ are the vertices corresponding to the edge $v_i v_{i+1}$ and $v_i u_i$ of $(P_n \odot K_1)$, $1 \le i \le n$. By the definition of middle graph the vertices $\{v_i^{'}, v_{i+1}^{'}, u_{i+1}^{'}, v_{i+1}\}$ induces a clique of order 4, hence $\chi_r(M(P_n \odot K_1)) \ge 4$. For $4 \le r \le \Delta$ by Lemma 3.1 $\chi_r(G) \ge \min\{r, \Delta(G)\} + 1$. This concludes the proof. □

**Theorem 3.4.** *Let $n \ge 4$, $M(P_n \odot K_1)$, then the $r$-dynamic chromatic number of the middle graph of a comb graph is*

$$\chi_r[M(P_n \odot K_1)] = \begin{cases} 4, & \text{for } 1 \le r \le 3 \\ r+1, & \text{for } 4 \le r \le \Delta \end{cases}.$$

*Proof.* The maximum and minimum degrees of $M(P_n \odot K_1)$ are $\Delta(M((P_n \odot K_1))) = 6$ and $\delta(M(P_n \odot K_1)) = 2$. Define the mapping $c : V \to Z^+$. We divide the proof into two cases.

**Case 1:** When $1 \leq r \leq 3$, by Lemma 3.3 the lower bound is $\chi_r(M((P_n \odot K_1))) \geq 4$. To show the upper bound, we use the following colorings:

- $c(v_1, v_2, \ldots, v_n) = \{3, 4, 3, 4, \ldots\}$
- $c(u_1^{'}, u_2^{'}, \ldots, u_n^{'}) = \{4, 3, 4, 3, \ldots\}$
- $c(u_1, u_2, \ldots, u_n) = 1$
- $c(v_1^{'}, v_2^{'}, \ldots, v_n^{'}) = \{2, 1, 2, 1, \ldots\}$

Thus we require 4 colors, that is $\chi_r(M(P_n \odot K_1)) \geq 4$. Hence $\chi_r(M(P_n \odot K_1)) = 4$.

**Case 2:** When $4 \leq r \leq \Delta$ , by Lemma 3.3 the lower bound is $\chi_r(M((P_n \odot K_1))) \geq 4$. To show the upper bound, we use the following colorings.

Subcase (i): when $r = 4$, consider the coloring:

- $c(v_1, v_2, \ldots, v_n) = \{1, 2, 3, 1, 2, 3, \ldots\}$
- $c(v_1^{'}, v_2^{'}, \ldots, v_n^{'}) = \{3, 1, 2, 3, 1, 2, \ldots\}$
- $c(u_1^{'}, u_2^{'}, \ldots, u_n^{'}) = \{r, r+1, r, r+1, \ldots\}$
- $c(u_1, u_2, \ldots, u_n) = \{r+1, r, r+1, r, \ldots\}$

Thus we require $r+1$ colors when $r = 4$.

Subcase (ii): When $r = 5$, consider the coloring:

- $c(v_1, v_2, \ldots, v_n) = \{2, 3, 2, 3, \ldots\}$
- $c(v_1^{'}, v_2^{'}, \ldots, v_n^{'}) = \{1, 4, 1, 4, \ldots\}$
- $c(u_1^{'}, u_2^{'}, \ldots, u_n^{'}) = \{r, r+1, r, r+1, \ldots\}$
- $c(u_1, u_2, \ldots, u_n) = \{r+1, r, r+1, r, \ldots\}$

Thus we require $r+1$ colors when $r = 5$.

Subcase (iii): When $r = 6$, consider the coloring:

- $c(v_1, v_2, \ldots, v_n) = \{4, 1, 5, 2, 3, 4, 1, 5, 2, 3, \ldots\}$
- $c(v_1^{'}, v_2^{'}, \ldots, v_n^{'}) = \{2, 3, 4, 1, 5, 2, 3, 4, 1, 5, \ldots\}$
- $c(u_1^{'}, u_2^{'}, \ldots, u_n^{'}) = \{r, r+1, r, r+1, \ldots\}$
- $c(u_1, u_2, \ldots, u_n) = \{r+1, r, r+1, r, \ldots\}$

Thus we require $r+1$ colors when $r = 6$.

Now from the subcases (i), (ii) and (iii) the $r$-adjacency condition is fulfilled. Therefore, $\chi_r(M(P_n \odot K_1)) \leq r+1$. Hence $\chi_r(M(P_n \odot K_1)) = r+1$. □

**Lemma 3.5.** *Let $n \geq 3$, $T(P_n \odot K_1)$ be the total graph of comb graph, then*

$$\chi_r(T(P_n \odot K_1)) \geq \begin{cases} 4, & \text{for } 1 \leq r \leq 3 \\ r+1, & \text{for } 4 \leq r \leq \Delta \end{cases}.$$

*Proof.* Let

$V(T(P_n \odot K_1)) = \{v_i : 1 \leq i \leq n\} \cup \{v_i^{'} : 1 \leq i \leq n-1\} \cup \{u_i : 1 \leq i \leq n\} \cup \{u_i^{'} : 1 \leq i \leq n\}$, where $v_i^{'}$ and $u_i^{'}$ are the vertices corresponding to the edge $v_i v_{i+1}$ and $v_i u_i$ of $(P_n \odot K_1)$, $1 \leq i \leq n$.

By the definition of total graph the vertices $\{v_i^{'}, v_{i+1}^{'}, u_{i+1}^{'}, v_{i+1}\}$ induce a clique of order 4, hence $\chi_r(T(P_n \odot K_1)) \geq 4$. For $4 \leq r \leq \Delta$ by Lemma 3.1 $\chi_r(G) \geq \min\{r, \Delta(G)\} + 1$. This concludes the proof. □

**Theorem 3.6.** *Let $n \geq 3$, $T(P_n \odot K_1)$ the $r$-dynamic chromatic number of the total graph of comb graph is*

$$\chi_r(T(P_n \odot K_1)) = \begin{cases} 4, & for\ 1 \leq r \leq 3 \\ r+1, & for\ 4 \leq r \leq \Delta \end{cases}$$

*Proof.* The maximum and minimum degrees of $T(P_n \odot K_1)$ are $\Delta(T(P_n \odot K_1)) = 6$ and $\delta(T(P_n \odot K_1)) = 2$. We divide the proof into two cases. Define the mapping $c : V \to Z^+$.

**Case 1:** When $1 \leq r \leq 3$, by Lemma 3.5 the lower bound is $\chi_r(T(P_n \odot K_1)) \geq 4$. To show the upper bound, we use the following colorings:

- $c(v_1, v_2, \ldots, v_n) = \{2, 3, 2, 3, \ldots\}$
- $c(u_1^{'}, u_2^{'}, \ldots, u_n^{'}) = \{3, 2, 3, \ldots\}$
- $c(u_1, u_2, \ldots, u_n) = \{1, 4, 1, 4, \ldots\}$
- $c(v_1^{'}, v_2^{'}, \ldots, v_n^{'}) = \{4, 1, 4, 1, \ldots\}$

Thus we require $4$ colors that is $\chi_r(T(P_n \odot K_1)) \leq 4$. Hence $\chi_r(T(P_n \odot K_1)) = 4$.

**Case 2:** When $4 \leq r \leq \Delta$, by Lemma 3.5 the lower bound is $\chi_r(T(P_n \odot K_1)) \geq 4$. To show the upper bound, we use the following colorings:

Subcase (i): When $r = 4$, consider the coloring:

- $c(v_1, v_2, \ldots, v_n) = \{1, 2, 3, 1, 2, 3, \ldots\}$
- $c(v_1^{'}, v_2^{'}, \ldots, v_n^{'}) = \{3, 1, 2, 3, 1, 2, \ldots\}$
- $c(u_1^{'}, u_2^{'}, \ldots, u_n^{'}) = \{r, r+1, r, r+1, \ldots\}$
- $c(u_1, u_2, \ldots, u_n) = \{r+1, r, r+1, r, \ldots\}$

Thus we require $r + 1$ colors when $r = 4$.

Subcase (ii): When $r = 5$, consider the coloring:

- $c(v_1, v_2, \ldots, v_n) = \{2, 3, 2, 3, \ldots\}$
- $c(v_1^{'}, v_2^{'}, \ldots, v_n^{'}) = \{1, 4, 1, 4, \ldots\}$
- $c(u_1^{'}, u_2^{'}, \ldots, u_n^{'}) = \{r, r+1, r, r+1, \ldots\}$
- $c(u_1, u_2, \ldots, u_n) = \{r+1, r, r+1, r, \ldots\}$

Thus we require $r + 1$ colors when $r = 5$.

Subcase (iii): When $r = 6$, consider the coloring:

- $c(v_1, v_2, \ldots, v_n) = \{4, 1, 5, 2, 3, 4, 1, 5, 2, 3, \ldots\}$
- $c(v_1^{'}, v_2^{'}, \ldots, v_n^{'}) = \{2, 3, 4, 1, 5, 2, 3, 4, 1, 5, \ldots\}$
- $c(u_1^{'}, u_2^{'}, \ldots, u_n^{'}) = \{r, r+1, r, r+1, \ldots\}$
- $c(u_1, u_2, \ldots, u_n) = \{r+1, r, r+1, r, \ldots\}$

Thus we require $r + 1$ colors when $r = 6$.

Now from the subcases (i), (ii) and (iii) the $r$-adjacency condition fulfilled. Therefore, $\chi_r(T(P_n \odot K_1)) \leq r + 1$. Hence $\chi_r(T(P_n \odot K_1)) = r + 1$. □

**Lemma 3.7.** *Let $n \geq 4$, $L(P_n \odot K_1)$ be the line graph of comb graph, then*

$$\chi_r(L(P_n \odot K_1)) \geq \begin{cases} 3, & \textit{for } 1 \leq r \leq 2 \\ 4, & \textit{for } r = 3 \\ 5, & \textit{for } r = 4 \end{cases}.$$

*Proof.* Let $V(L(P_n \odot K_1)) = e_1, e_2, e_3, \ldots, e_n, e_1^{'}, e_2^{'}, e_3^{'}, \ldots, e_{n-1}^{'}$. The maximum and minimum degrees of $L(P_n \odot K_1)$ are $\Delta(L(P_n \odot K_1)) = 4$ and $\delta(L(P_n \odot K_1)) = 1$. For $1 \leq r \leq 2$ by the definition of line graph the vertices $e_i, e_i^{'}, e_{i+1}^{'}$ induces a clique of order 3. Hence $\chi_r(L(P_n \odot K_1)) \geq 3$.

For $r = 3$ by Lemma 3.1

$$\chi_r(L(P_n \odot K_1)) \geq \min\{r, \Delta(L(P_n \odot K_1)\} \geq \min\{3, \Delta(L(P_n \odot K_1))\} + 1 = 3 + 1 = 4.$$

For $r = 4$ by Lemma 3.1

$$\chi_r(L(P_n \odot K_1)) \geq \min\{r, \Delta(L(P_n \odot K_1))\} \geq \min\{4, \Delta(L(P_n \odot K_1))\} + 1 = 4 + 1 = 5.$$

This concludes the proof. □

**Theorem 3.8.** *Let $n \geq 4$, $L(P_n \odot K_1)$ the $r$-dynamic chromatic number of a line graph of comb graph is*

$$\chi_r(L(P_n \odot K_1)) = \begin{cases} 3, & \textit{for } 1 \leq r \leq 2 \\ 4, & \textit{for } r = 3 \\ 5, & \textit{for } r = 4 \end{cases}.$$

*Proof.* Define the mapping $c : V \rightarrow Z^+$. We divide the proof into three cases.

**Case 1:** For $1 \leq r \leq 2$. By the Lemma 3.7 the lower bound is $\chi_r(L(P_n \odot K_1)) \geq 3$. To show the upper bound, we use the following colorings:

- $c(e_1, e_2, \ldots, e_n) = 1$
- $c(e_1^{'}, e_2^{'}, \ldots, e_n^{'}) = \{2, 3, 2, 3, \ldots\}$

Thus we require 3 colors, that is $\chi_r(L(P_n \odot K_1)) \leq 3$. Hence $\chi_r(L(P_n \odot K_1)) = 3$ when $1 \leq r \leq 2$.

**Case 2:** For $r = 3$. By the Lemma 3.7 the lower bound is $\chi_r(L(P_n \odot K_1)) \geq 4$. The $r$-dynamic coloring are as follows:

- $c(e_1, e_2, \ldots, e_n) = \{1, 3, 1, 3, \ldots\}$
- $c(e_1^{'}, e_2^{'}, \ldots, e_n^{'}) = \{2, 4, 2, 4, \ldots\}$

Thus we require 4 colors, that is $\chi_r(L(P_n \odot K_1)) \leq 4$. Hence $\chi_r(L(P_n \odot K_1)) = 4$ when $r = 3$.

**Case 3:** For $r = 4$. By the Lemma 3.7 the lower bound is $\chi_r(L(P_n \odot K_1)) \geq 5$. The $r$-dynamic coloring are as follows:

- $c(e_1, e_2, \ldots, e_n) = \{1, 3, 1, 3, \ldots\}$
- $c(e_1^{'}, e_2^{'}, \ldots, e_n^{'}) = \{2, 4, 5, 2, 4, 5, \ldots\}$

Thus we require 5 colors, that is $\chi_r(L(P_n \odot K_1)) \leq 5$. Hence $\chi_r(L(P_n \odot K_1)) = 5$ when $r = 4$. □

**Lemma 3.9.** *Let $n \geq 3$, $P(P_n \odot K_1)$ be the para-line graph of comb graph then*

$$\chi_r(P(P_n \odot K_1)) \geq \begin{cases} 3, & \text{for } 1 \leq r \leq 2 \\ 4, & \text{for } r = \Delta \end{cases}.$$

*Proof.* Let $V(P(P_n \odot K_1)) = \{e_i; 1 \leq i \leq 2n\} \cup \{e_i^{'}; 1 \leq i \leq 2n-2\}$, where $e_i$ is the vertex corresponding to the edge $v_iv_{i+1}$ and $e_i^{'}$ is the vertex corresponding to the edge $v_iu_i$ of $(P_n \odot k_1)$, $1 \leq i \leq n$. For $1 \leq r \leq 2$, by the definition of para-line graph the vertices $e_{2i+2}, e_{i+1}^{'}, e_{i+2}^{'}$ induces a clique of order 3. Hence $\chi_r(P(P_n \odot K_1)) \geq 3$.

For $r = \Delta$ by Lemma 3.1 $\chi_r(L(P_n \odot K_1)) \geq \min\{r, \Delta(P(P_n \odot K_1))\} + 1 = 3 + 1 = 4$. Therefore $\chi_r(L(P_n \odot K_1)) \geq 4$. This concludes the proof. □

**Theorem 3.10.** *Let $n \geq 3$, $P(P_n \odot K_1)$ the r-dynamic chromatic number of para-line graph of a comb graph is*

$$\chi_r(P(P_n \odot K_1)) = \begin{cases} 3, & \text{for } 1 \leq r \leq 2 \\ 4, & \text{for } r = \Delta \end{cases}.$$

*Proof.* The maximum and minimum degrees of $L(P_n \odot K_1)$ are $\Delta(L(P_n \odot K_1)) = 3$ and $\delta(L(P_n \odot K_1)) = 1$. Define the mapping $c : V \to Z^+$.

**Case 1:** For $1 \leq r \leq 2$. By the Lemma 3.9 the lower bound is $\chi_r(L(P_n \odot K_1)) \geq 3$. To show the upper bound, we assign colors as follows:

- $c(e_1, e_2, \ldots, e_n) = \{1, 2, 1, 2, \ldots\}$
- $c(e_1^{'}, e_2^{'}, \ldots, e_n^{'}) = \{1, 3, 1, 3, \ldots\}$

Thus we require 3 colors, that is $\chi_r(P(P_n \odot K_1)) \leq 3$. Hence $\chi_r(P(P_n \odot K_1)) = 3$ when $1 \leq r \leq 2$.

**Case 2:** For $r = 3$. By the Lemma 3.9 the lower bound is $\chi_r(P(P_n \odot K_1)) \geq 4$. To show the upper bound, we assign colors as follows:

- $c(e_1, e_2, \ldots, e_{2n-1}) = \{3, 1, 3, 1, \ldots\}$ and $c(e_{2n}) = 4$
- $c(e_1^{'}, e_2^{'}, \ldots, e_n^{'}) = \{1, 2, 1, 2, \ldots\}$

Thus we require 4 colors, that is $\chi_r(P(P_n \odot K_1)) \leq 4$. Hence $\chi_r(P(P_n \odot K_1)) = 4$ when $r = 3$. □

**Theorem 3.11.** *Let $n \geq 3$, and $S(P_n \odot K_1)$ be the sub-division graph of $(P_n \odot K_1)$, then*

$$\chi_r(S(P_n \odot K_1)) = r + 1, 1 \leq r \leq 3$$

*Proof.* Let $V(S(P_n \odot K_1)) = \{v_i : 1 \leq i \leq n\} \cup \{v_i^{'} : 1 \leq i \leq n-1\} \cup \{u_i : 1 \leq i \leq n\} \cup \{u_i^{'} : 1 \leq i \leq n\}$ where $v_i^{'}$ and $u_i^{'}$ are the vertex corresponding to the edge $v_iv_{i+1}$ and $v_iu_i$ of $(P_n \odot K_1), 1 \leq i \leq n$. The maximum and minimum degrees of $S(P_n \odot K_1)$ are $\Delta(S(P_n \odot K_1)) = 3$ and $\delta(S(P_n \odot K_1)) = 1$. By Lemma 3.1 $\chi_r(S(P_n \odot K_1)) \geq \min\{r, \Delta(S(P_n \odot K_1))\} + 1 = r + 1$, for $1 \leq r \leq 3$. To show the upper bound, we assign colors as follows:

**Case 1:** When $r = 1$,

- $c(v_1, v_2, \ldots, v_n) = 1$
- $c(v_1^{'}, v_2^{'}, \ldots, v_n^{'}) = 2$
- $c(u_1^{'}, u_2^{'}, \ldots, u_n^{'}) = 2$
- $c(u_1, u_2, \ldots, u_n) = 1$

Thus we require 2 colors, that is $\chi_r(S(P_n \odot K_1)) \leq 2$. Hence $\chi_r(S(P_n \odot K_1)) = 2$ when $r = 1$.

**Case 2:** When $r = 2$,

- $c(v_1, v_2, \ldots, v_n) = \{1, 2, 1, 2, \ldots\}$
- $c(v_1^{'}, v_2^{'}, \ldots, v_n^{'}) = 3$
- $c(u_1^{'}, u_2^{'}, \ldots, u_n^{'}) = \{2, 1, 2, 1, \ldots\}$
- $c(u_1, u_2, \ldots, u_n) = 3$

Thus we require 3 colors, that is $\chi_r(S(P_n \odot K_1)) \leq 3$. Hence $\chi_r(S(P_n \odot K_1)) = 3$ when $r = 2$.

**Case 3:** When $r = 3$,

- $c(v_1, v_2, \ldots, v_n) = \{1, 2, 1, 2, \ldots\}$
- $c(v_1^{'}, v_2^{'}, \ldots, v_n^{'}) = \{3, 4, 3, 4, \ldots\}$
- $c(u_1^{'}, u_2^{'}, \ldots, u_n^{'}) = \{2, 1, 2, 1, \ldots\}$
- $c(u_1, u_2, \ldots, u_n) = 4$

Thus we require 4 colors, that is $\chi_r(S(P_n \odot K_1)) \leq 4$. Hence $\chi_r(S(P_n \odot K_1)) = 4$ when $r = 3$. Therefore $\chi_r(S(P_n \odot K_1)) \leq r + 1, 1 \leq r \leq 3$. Hence $\chi_r(S(P_n \odot K_1)) = r + 1$, $1 \leq r \leq 3$. □

# 4 Conclusion

In this paper we have investigated the $r$-dynamic chromatic number of some operations such as central graph, middle graph, total graph, line graph, para-line graph, and subdivision graph of the comb graph $P_n \odot K_1$. We have used the upper bound and lower bound method to obtain the $r$-dynamic chromatic number. Let $G$ be any connected graph, till now there is no sharp lower bound for the $r$-dynamic chromatic number, so it is left as an open problem.


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