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Generalized Lucas numbers of the form 3×2^m

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Abstract: For an integer $k \geq 2$, let $(L_n^{(k)})_n$ be the k-generalized Lucas sequence which starts with $0, \ldots, 0, 2, 1$ (k terms) and each term afterwards is the sum of the k preceding terms. In this paper, we look the k-generalized Lucas numbers of the form 3×2^m i.e. we study the Diophantine equation $L_n^{(k)} = 3 \times 2^m$ in positive integers n, k, m with $k \geq 2$.

Keywords: k-generalized Lucas numbers, Linear form in logarithms, Reduction method.
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1 Introduction

Let $k \geq 2$ be an integer. We consider a generalization of Lucas sequence called the k-generalized Lucas sequence $L_n^{(k)}$ defined as

$$L_n^{(k)} = L_{n-1}^{(k)} + L_{n-2}^{(k)} + \cdots + L_{n-k}^{(k)} \quad \text{for all } n \geq 2, \tag{1}$$

with the initial conditions $L_{-(k-2)}^{(k)} = L_{-(k-3)}^{(k)} = \cdots L_{-1}^{(k)} = 0$, $L_0^{(k)} = 2$ and $L_1^{(k)} = 1$. If $k = 2$, we obtain the classical Lucas sequence

$$L_0 = 2, \quad L_1 = 1, \quad \text{and} \quad L_n = L_{n-1} + L_{n-2} \quad \text{for} \quad n \geq 2.$$

$$(L_n)_{n\geq 0} = \{2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, \ldots\}.$$

If $k = 3$, then the 3-Lucas sequence is

$$(L_n^{(3)})_{n\geq -1} = \{0, 2, 1, 3, 6, 10, 19, 35, 64, 118, 217, 399, 734, 1350, 2483, 4567, \ldots\}.$$

If $k = 4$, then the 4-Lucas sequence is

$$(L_n^{(4)})_{n\geq -2} = \{0, 0, 2, 1, 3, 6, 12, 22, 43, 83, 160, 308, 594, 1145, 2207, 4254, 8200, \ldots\}.$$

It is known that if $2 \leq n \leq k$, then

$$L_n^{(k)} = 3 \times 2^{n-2}, \tag{2}$$

see Lemma 2 in [5]. This raises the following natural question: *are there any positive integers n, m, k such that*

$$L_n^{(k)} = 3 \times 2^m? \tag{3}$$

The aim of this paper is to give an answer to this problem by proving the following result.

Theorem 1.1. *The Diophantine equation* (3) *has no solution if* $n \geq k + 1$.

Our proof of Theorem 1.1 is mainly based on linear forms in logarithms of algebraic numbers and a reduction algorithm originally introduced by Baker and Davenport [2]. Here, we use a version due to Dujella and Pethő in [6, Lemma 5(a)].

2 The tools

2.1 Linear forms in logarithms

For any non-zero algebraic number η of degree d over $\mathbb{Q}$, whose minimal polynomial over $\mathbb{Z}$ is $a\prod_{j=1}^{d}\left(X - \eta^{(j)}\right)$, we denote by

$$h(\eta) = \frac{1}{d}\left(\log|a| + \sum_{j=1}^{d}\log\max\left(1, |\eta^{(j)}|\right)\right)$$

the usual absolute logarithmic height of η. In particular, if $\eta = p/q$ is a rational number with $\gcd(p, q) = 1$ and $q > 0$, then $h(\eta) = \log\max\{|p|, q\}$. The following properties of the function absolute logarithmic height $h()$, which will be used in the next sections without special reference, are also known:

$$h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log 2, \tag{4}$$

$$h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma), \tag{5}$$

$$h(\eta^s) = |s|\, h(\eta) \quad (s \in \mathbb{Z}). \tag{6}$$

With this notation, Matveev proved the following theorem (see [7]).

Theorem 2.1. *Let $\eta_1, \ldots, \eta_s$ be real algebraic numbers and let $b_1, \ldots, b_s$ be nonzero rational integer numbers. Let $d_{\mathbb{K}}$ be the degree of the number field $\mathbb{Q}(\eta_1, \ldots, \eta_s)$ over $\mathbb{Q}$ and let A_j be a positive real number satisfying*

$$A_j = \max\{d_{\mathbb{K}} h(\eta), |\log \eta|, 0.16\} \quad \textit{for} \quad j = 1, \ldots, s.$$

Assume that $B \geq \max\{|b_1|, \ldots, |b_s|\}$. If $\eta_1^{b_1} \cdots \eta_s^{b_s} - 1 \neq 0$, then

$$|\eta_1^{b_1} \cdots \eta_s^{b_s} - 1| \geq \exp(-1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot d_{\mathbb{K}}^2 (1 + \log d_{\mathbb{K}})(1 + \log B) A_1 \cdots A_s).$$

2.2 Reduction algorithm

The following lemma can be found in [1].

Lemma 2.2. *Let M be a positive integer, p/q be a convergent of the continued fraction of the irrational γ such that $q > 6M$, and let A, C, μ be some real numbers with $A > 0$ and $C > 1$. Let*

$$\varepsilon = ||\mu q|| - M \cdot ||\gamma q||,$$

where $|| \cdot ||$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no solution of the inequality

$$0 < u\gamma - v + \mu < AC^{-w}$$

in positive integers u, v and w with

$$u \leq M \quad \textit{and} \quad w \geq \frac{\log(Aq/\varepsilon)}{\log C}.$$

2.3 Properties of the *k*-generalized Lucas sequence

In this subsection, we recall some facts and properties of these sequences which will be used later.

We know that the characteristic polynomial of the k-generalized Lucas numbers $(L_n^{(k)})_n$, namely

$$\Psi_k(x) = x^k - x^{k-1} - \cdots - x - 1,$$

is irreducible over $\mathbb{Q}[x]$ and has just one root outside the unit circle; the other roots are strictly inside the unit circle (see, for example, [8,9,12]). We denote by $\alpha := \alpha(k)$ the single root, which is located between $2(1 - 2^{-k})$ and 2 (see [12]). We label its roots by $\alpha_1, \ldots, \alpha_k$ with $\alpha := \alpha_1$. To simplify the notation, in general, we omit the dependence on k of α.

For an integer $s \geq 2$, we define the function

$$f_s(x) = \frac{x - 1}{2 + (s + 1)(x - 2)}. \tag{7}$$

Now, we are ready to recall in the following lemmas some properties of the sequence $(L_n^{(k)})_{n \geq -(k-2)}$, which will be used for the proof of Theorem 1.1.

Lemma 2.3. [5, p. 144]

(a) For all $n \geq 1$ and $k \geq 2$, we have

$$\alpha^{n-1} \leq L_n^{(k)} \leq 2\alpha^n. \tag{8}$$

(b) The following "Binet-like" formula holds for all $n \geq -(k-2)$:

$$L_n^{(k)} = \sum_{i=1}^{k} (2\alpha_i - 1) f_k(\alpha_i) \alpha_i^{n-1}. \tag{9}$$

(c) For all $n \geq -(k-2)$, we have

$$\left| L_n^{(k)} - (2\alpha - 1) f_k(\alpha) \alpha^{n-1} \right| < \frac{3}{2}. \tag{10}$$

Lemma 2.4. [11] *For every positive integer $n \geq 2$, we have*

$$L_n^{(k)} \leq 3 \cdot 2^{n-2}. \tag{11}$$

Moreover, if $n \geq k+2$, then the above inequality is strict.

Lemma 2.5. [4, pp. 89–90] *For $k \geq 2$, let α be the dominant root of $\Psi_k(x)$, and consider the function $f_s(x)$ defined in* (7)*. Then:*

(i) The inequalities $1/2 < f_k(\alpha) < 3/4$ and $\left| f_k(\alpha^{(i)}) \right| < 1$, for $2 \leq i \leq k$ hold. So the number $f_k(\alpha)$ is not an algebraic integer.

(ii) The logarithmic height function satisfies $h(f_k(\alpha)) < 3 \log k$.

3 The proof of the main result

3.1 An inequality for n and m in terms of k

From now on, we assume that $n \geq k+1$. By Lemma 2.4 and Equation (3) we get

$$3 \cdot 2^m = L_n^{(k)} \leq 3 \cdot 2^{n-2},$$

so we deduce that $m < n-1$. Thus, we may suppose that $n \geq 3$ and $m \geq 2$.

Now, we prove the following lemma.

Lemma 3.1. *If (n, k, m) is a nontrivial solution in integers of Equation* (3) *with $k \geq 2$ and $n \geq k+1$, then the inequalities*

$$m \leq n < 4 \times 10^{14} k^4 \log^3 k \tag{12}$$

hold.

Proof. Combining (3) with (10), one gets

$$\left| 3 \times 2^m - (2\alpha - 1) f_k(\alpha) \alpha^{n-1} \right| < \frac{3}{2}. \tag{13}$$

Notice that $\alpha > 1$, $2^k > k+1$ and $2^k > (k+1)(2 - (2 - 2^{-k+1})) > (k+1)(2-\alpha)$.

Thus, $(2\alpha - 1) f_k(\alpha) \alpha^{n-1}$ is positive. Now, we divide both sides of the above inequality by $(2\alpha - 1) f_k(\alpha) \alpha^{n-1}$ to obtain the following inequality

$$\left| 3 \cdot 2^m \cdot \alpha^{-(n-1)} \cdot ((2\alpha - 1) f_k(\alpha))^{-1} - 1 \right| < \frac{3}{\alpha^{n-1}}, \tag{14}$$

where we used the facts $2 + (k+1)(\alpha - 2) < 2$ and $1/(2\alpha - 1) < 1/2$.

In order to prove inequalities (12), we will apply Theorem 2.1. To this end, we take

$$t := 3, \quad \eta_1 := 2, \quad \eta_2 := \alpha, \quad \eta_3 := 3((2\alpha-1)f_k(\alpha))^{-1},$$

and

$$b_1 := m, \quad b_2 := -(n-1), \quad b_3 := 1.$$

We put

$$\Lambda := 3 \cdot 2^m \cdot \alpha^{-(n-1)} \cdot ((2\alpha-1)f_k(\alpha))^{-1} - 1 \tag{15}$$

and inequality (14) becomes

$$|\Lambda| < \frac{3}{\alpha^{n-1}}. \tag{16}$$

For these choices, the field $\mathbb{K} := \mathbb{Q}(\alpha)$ contains η_1, η_2, η_3 and has $d_{\mathbb{K}} = k$. Since $h(\eta_1) = \log 2$ and $h(\eta_2) = (\log \alpha)/k < (\log 2)/k$, it follows that

$$\max\{kh(\eta_1), |\log \eta_1|, 0.16\} = k \log 2 := A_1$$

and

$$\max\{kh(\eta_2), |\log \eta_2|, 0.16\} = \log 2 := A_2.$$

On the other hand, we use the estimate (ii) of Lemma 2.5 and the properties (5), (6) to deduce that for all $k \geq 2$

$$\begin{array}{rcl} h(\eta_3) & \leq & h(2\alpha-1) + h(f_k(\alpha)) + h(3) \\ & < & \log 3 + 3\log k + \log 3 \\ & < & 7 \log k, \end{array}$$

so we get

$$\max\{kh(\eta_3), |\log \eta_3|, 0.16\} = 7k \log k := A_3.$$

As $m < n-1$, we can take $B := n-1$.

Before applying Theorem 2.1, it remains us to prove that $\Lambda \neq 0$. Assume the contrary, i.e., $\Lambda = 0$, this imply that

$$3 \cdot 2^m = \frac{(2\alpha-1)(\alpha-1)}{2+(k+1)(\alpha-2)}\alpha^{n-1}.$$

If we conjugate the above relation by the automorphism of Galois $\sigma : \alpha \to \alpha_i$ $(i > 1)$ and then taking absolute values, we get

$$3 \cdot 2^m = \left| \frac{(2\alpha_i-1)(\alpha_i-1)}{2+(k+1)(\alpha_i-2)}\alpha_i^{n-1} \right|.$$

But the above relation is not possible since its left-hand side is greater than or equal to 12, while its right-hand side is smaller than $6/(k-1) < 8$ as $|\alpha_i| < 1$ and

$$|2+(k+1)(\alpha_i-2)| \geq (k+1)|\alpha_i - 2| - 2 > k-1.$$

Thus, $\Lambda \neq 0$. Therefore, applying Theorem 2.1 to get a lower bound for $|\Lambda|$ and comparing this with inequality (16), we get

$$(n-1)\log \alpha - \log 3 < 4.82 \times 10^{11} k^4 \log k(1+\log k)(1+\log(n-1)).$$

Since $1 + \log k < 2 \log k$, $1 + \log(n-1) < 2\log(n-1)$ and $1/\log \alpha < 2$ for $k \geq 3$ and $n \geq 4$, we conclude that

$$\frac{n-1}{\log(n-1)} < 4 \times 10^{12} k^4 \log^2 k. \tag{17}$$

We know that the function $x \mapsto x/\log x$ is increasing for all $x > e$, so it is easy to check that the inequality $\dfrac{x}{\log x} < A$ implies $x < 2A\log A$, whenever $A \geq 3$.

Thus, taking $A := 4\times 10^{12}k^4\log^2 k$, inequality (17) and as $30+4\log k+2\log\log k < 47\log k$, for all $k \geq 3$, we get

$$\begin{aligned} n-1 &< 2(4\times 10^{12}k^4\log^2 k)\log(4\times 10^{12}k^4\log^2 k) \\ &< (8\times 10^{12}k^4\log^2 k)(30+4\log k+2\log\log k) \\ &< 4\times 10^{14}k^4\log^3 k. \end{aligned}$$

□

3.2 The case $2 \leq k \leq 170$

In this subsection, we study the cases when $k \in [2, 170]$. We prove the following lemma.

Lemma 3.2. *The Diophantine equation* (3) *has no solution when* $k \in [2, 170]$ *and* $n \geq k+1$.

Proof. Let

$$\Gamma = m\log 2 - (n-1)\log\alpha - \log\left(\frac{(2\alpha-1)f_k(\alpha)}{3}\right). \tag{18}$$

Then $e^\Gamma - 1 = \Lambda$, where Λ is defined by (15). Therefore, (16) can be rewritten as

$$\left|e^\Gamma - 1\right| < \frac{3}{\alpha^{n-1}}. \tag{19}$$

Notice that $\Gamma \neq 0$ since $\Lambda \neq 0$, so we distinguish the following two cases.

- First, if $\Gamma > 0$, then $e^\Gamma - 1 > 0$. Using the fact that $x \leq e^x - 1$ for all $x \in \mathbb{R}$, inequality (19) gives

  $$0 < \Gamma < \frac{3}{\alpha^{n-1}}.$$

  Replacing Γ in the above inequality by its formula (18), dividing both sides of the resulting inequality by $\log\alpha$ and using the fact that $1/\log\alpha < 2$ for all $k \geq 2$, we get

  $$0 < m\left(\frac{\log 2}{\log\alpha}\right) - n + \left(1 - \frac{\log((2\alpha-1)f_k(\alpha)/3)}{\log\alpha}\right) < 6\cdot\alpha^{-(n-1)}. \tag{20}$$

  Putting

  $$\gamma := \frac{\log 2}{\log\alpha}, \quad \mu := 1 - \frac{\log((2\alpha-1)f_k(\alpha)/3)}{\log\alpha}, \quad A := 6, \quad \text{and} \quad C := \alpha,$$

  the above inequality (20) yields

  $$0 < m\gamma - n + \mu < AC^{-(n-1)}. \tag{21}$$

  It is clear that γ is an irrational number because $\alpha > 1$ is a unit in $\mathcal{O}_{\mathbb{K}}$, the ring of integers of $\mathbb{K}$. So α and 2 are multiplicatively independent.

  For each $k \in [2, 170]$, we find a good approximation of α and a convergent p_ℓ/q_ℓ of the continued fraction of γ such that $q_\ell > 6M$, where $M = \lfloor 4\times 10^{14}k^4\log^3 k\rfloor$, which is an upper bound on m from Lemma 12. After doing this, we use Lemma 2.2 to Inequality (20). A computer search with Mathematica revealed that the maximum value of $\lfloor\log(Aq/\varepsilon)/\log C\rfloor$ over all $k \in [2, 170]$ is 175, which according to Lemma 2.2, is an upper bound on $n-1$. Hence, we deduce that the possible solutions (n, k, m) of Equation (3) for which $k \in [2, 170]$ and $\Gamma > 0$ have $n \leq 176$, therefore $m \leq 175$, since $m < n$.

- Now, we consider the case $\Gamma < 0$. It is easy to see that $2/\alpha^{n-1} < 1/2$ holds for all $k \geq 2$ and $n \geq 3$. Thus, inequality (19) implies $\left|e^{\Gamma} - 1\right| < 1/2$ and therefore $e^{|\Gamma|} < 2$. As $\Gamma < 0$, we get $0 < |\Gamma| \leq e^{|\Gamma|} - 1 = e^{|\Gamma|}\left|e^{\Gamma} - 1\right| < \dfrac{6}{\alpha^{n-1}}$. Similarly, as the case when $\Gamma > 0$, we get
$$0 < (n-1)\gamma - m + \mu < AC^{-(n-1)}, \tag{22}$$
where
$$\gamma := \frac{\log \alpha}{\log 2}, \quad \mu := \frac{\log((2\alpha - 1)f_k(\alpha)/3)}{\log 2}, \quad A := 9, \quad C := \alpha.$$
In this case, we also took $M := \lfloor 4 \times 10^{14} k^4 \log^3 k \rfloor$ which is an upper bound of $n-1$ by Lemma 3.1, and we applied Lemma 2.2 to inequality (22). In this case, with the help of Mathematica, we found that the maximum value of $\lfloor \log(Aq/\varepsilon)/\log C \rfloor$ is 174. Thus, the possible solutions (n, k, m) of Equation (3) in the range $k \in [2, 170]$ and $\Gamma < 0$ give $n \leq 175$, so $m \leq 174$.

Finally, using Mathematica we compared $L_n^{(k)}$ and 3×2^m for the range $3 \leq n \leq 175$ and $2 \leq m \leq 174$, with $m < n$ and found that Equation (3) has no solution in this range. □

3.3 The case $k > 170$

In this subsection, we analyze the case $k > 170$.

Lemma 3.3. *The Diophantine equation* (3) *has no solution when* $k > 170$ *and* $n \geq k+1$.

Proof. For $k > 170$, we have $n < 4 \times 10^{14} k^4 \log^3 k < 2^{k/2}$. In [5, p. 150], it was proved that
$$(2\alpha - 1)f_k(\alpha)\alpha^{n-1} = 3 \cdot 2^{n-2} + 3 \cdot 2^{n-1}\eta + \frac{\delta}{2} + \eta\delta,$$
where
$$|\eta| < \frac{2k}{2^k} \quad \text{and} \quad |\delta| < \frac{2^{n+2}}{2^{k/2}}.$$
Thus, from the above equality and (13), we get
$$\begin{aligned} |3 \cdot 2^m - 3 \cdot 2^{n-2}| &= \left|3 \cdot 2^m - (2\alpha - 1)f_k(\alpha)\alpha^{n-1} + 3 \cdot 2^{n-1}\eta + \frac{\delta}{2} + \eta\delta\right| \\ &< \frac{3}{2} + \frac{3k \cdot 2^n}{2^k} + \frac{2^{n+1}}{2^{k/2}} + \frac{2^{n+3}k}{2^{3k/2}}. \end{aligned}$$
For $k > 170$, we get $4k/2^k < 1/2^{k/2}$, $8/(3 \cdot 2^{k/2}) < 3/2^{k/2}$ and $32k/(3 \cdot 2^{3k/2}) < 1/2^{k/2}$. Thus, we obtain
$$\left|3 \cdot 2^m - 3 \cdot 2^{n-2}\right| < 18 \cdot \frac{2^{n-2}}{2^{k/2}}.$$
As $n \geq k+1$, we have $1/2^{n-1} < 1/2^{k/2}$ and factoring out $3 \cdot 2^{n-2}$ in the right-hand side of the above inequality, we get
$$\left|2^{m-n+2} - 1\right| < \frac{6}{2^{k/2}}. \tag{23}$$
Moreover, since $m < n$, we have that $m - n + 2 \leq 1$, then it follows from (23) that
$$\frac{1}{2} < 1 - 2^{m-n+2} < \frac{6}{2^{k/2}}.$$

So, $2^{k/2} < 12$, which is impossible as $k > 170$. Hence, we have shown that there are no solutions (n, k, m) to Equation (3) with $k > 170$. $\square$

Thus, this completes the proof of Theorem 1.1.

4 Conclusion

In this paper, we prove that there are no positive integers m, n, k such that a k-generalized Lucas number has the form 3×2^m for $n \geq k+1$, i.e., the Diophantine equation $L_n^{(k)} = 3 \times 2^m$ has no solution in positive integers n, k, m with $k \geq 2$ and $n \geq k+1$.

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