Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 27, 2021, No. 2, 88–100
DOI: 10.7546/nntdm.2021.27.2.88-100

On the Diophantine equations $z^2 = f(x)^2 \pm f(x)f(y) + f(y)^2$

Qiongzhi Tang

School of Mathematics and Statistics, Changsha University of Science and Technology,
Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering
Changsha, 410114, China
e-mail: tangqiongzhi628@163.com

Received: 22 August 2020 **Revised:** 31 March 2021 **Accepted:** 8 April 2021

Abstract: Using the theory of Pell equation, we study the non-trivial positive integer solutions of the Diophantine equations $z^2 = f(x)^2 \pm f(x)f(y) + f(y)^2$ for certain polynomials $f(x)$, which mean to construct integral triangles with two sides given by the values of polynomials $f(x)$ and $f(y)$ with the intersection angle $120°$ or $60°$.
Keywords: Diophantine equation, Pell equation, Positive integer solution.
2020 Mathematics Subject Classification: 11D25 (primary), 11D72 (secondary).

1 Introduction

Let $f(x) \in \mathbb{Q}(x)$ be a polynomial without multiple roots and with $\deg f(x) \geq 2$. Many authors studied the non-trivial integer or rational (parametric) solutions of the Diophantine equations

$$z^2 = f(x)^2 + f(y)^2 \tag{1}$$

and

$$z^2 = f(x)^2 - f(y)^2. \tag{2}$$

The positive integer or rational solutions of Eqs. (1) and (2) mean to form right triangles with two legs given by the values of polynomials $f(x)$ and $f(y)$, or one leg given by $f(y)$ and the hypotenuse given by $f(x)$. Let us recall that the solution (x, y, z) is a non-trivial solution of Eq. (1) [respectively, Eq. (2)] if $f(x)f(y) \neq 0$ [respectively, $f(x) \neq |f(y)|$ and $f(y) \neq 0$].

In 1962, W. Sierpiński [8] obtained infinitely many non-trivial positive integer solutions of Eq. (1) for $f(x) = \frac{x(x+1)}{2}$. In 2010, M. Ulas and A. Togbé [10] studied the non-trivial rational (parametric) solutions of Eqs. (1) and (2) for some quadratic and cubic polynomials. In the same year, B. He, A. Togbé and M. Ulas [4] investigated the non-trivial positive integer solutions of Eq. (1) for $f(x) = \frac{x(x+1)}{2}, \frac{x(x+1)(x+2)}{6}$ and the non-trivial positive integer solutions of Eq. (2) for $f(x) = x^2 + a$. In 2018, Y. Zhang and A. S. Zargar [14] proved that Eq. (1) has infinitely many non-trivial rational solutions for $f(x) = x(x-1)(x+1)(x+\frac{1-k^2}{2k})$, $k \in \mathbb{Z}\backslash\{0, \pm 1\}$, and similar result was obtained for Eq. (2) when $f(x) = x(x-1)(x+1)(x-\frac{2k}{k^2+1})$, $k \in \mathbb{Z}\backslash\{0, \pm 1\}$, which gives a positive answer to Question 4.3 of [10] for quartic polynomials. In 2019, A. E. A. Youmbai and D. Behloul [11] extended the results of [14] to the polynomials $f(x) = x\prod_{t=0}^{n}(x-k^t)(x+k^t)$ of degree $2n+3$ and gave a positive answer to Question 4.3 of [10] for the polynomials $f(x) = x\prod_{t=0}^{n}(x+k^t)$ of degree $n+2$. In 2021, Y. Zhang and Q. Z. Tang [12] showed that Eqs. (1) and (2) have infinitely many non-trivial integer solutions for polynomials $f(x)$ with integer coefficients and degree n.

In 1783, L. Euler [2] studied the non-trivial rational solutions of Eq.(1) for $f(x) = x+\frac{1}{x}$. In 2019, Y. Zhang and A. S. Zargar [15] investigated the non-trivial rational (parametric) solutions of Eqs. (1) and (2) for some simple Laurent polynomials $f(x)$, such as $f(x) = x + b + \frac{c}{x}$, $\frac{(x+1)(x+b)(x+c)}{x}$ with non-zero integers b and c. In the same year, Y. Zhang, Q. Z. Tang and Y. N. Zhang [13] got the conditions for $f(x) = b + \frac{c}{x}$ with non-zero integers b and c such that Eqs. (1) and (2) have infinitely many non-trivial solutions $x, y \in \mathbb{Z}$ and $z \in \mathbb{Q}$, which gave a positive answer to Question 3.2 of [15]. Meanwhile, they [13] studied the non-trivial rational solutions of Eqs. (1) and (2) for Laurent polynomials

$$f(x) = \frac{\prod_{t=0}^{n}(x+k^t)}{x}, \frac{\prod_{t=0}^{n}(x-k^t)(x+k^t)}{x},$$

$n \geq 1$, $k \in \mathbb{Z}\backslash\{0, \pm 1\}$, and gave a positive answer to Question 3.1 of [15].

2 Main results

In this paper, we consider the non-trivial positive integer solutions of the Diophantine equations

$$z^2 = f(x)^2 + f(x)f(y) + f(y)^2 \tag{3}$$

and

$$z^2 = f(x)^2 - f(x)f(y) + f(y)^2. \tag{4}$$

The positive integer solutions of Eqs. (3) and (4) mean to construct integral triangles with two sides given by the values of polynomials $f(x)$ and $f(y)$ with the intersection angle $120°$ or $60°$ (see Figure 1), which are not Heron triangles. Let us recall that an integral triangle is a triangle with integral sides, and a Heron triangle is a triangle with integral sides and integral area.

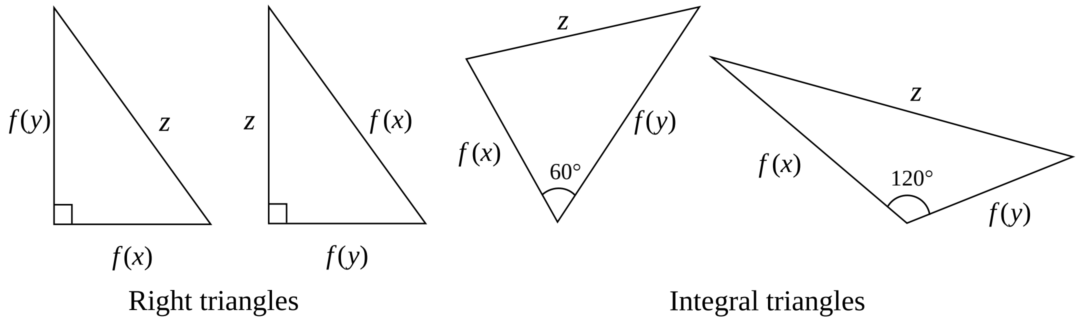

Figure 1

Several authors [1, 3, 6, 7] studied the integral triangles with an intersection angle $120°$ or $60°$. When $f(x) = x^2$, Eqs. (3) and (4) are

$$z^2 = x^4 + x^2y^2 + y^4$$

and

$$z^2 = x^4 - x^2y^2 + y^4,$$

which have no non-trivial positive integer solutions, we can refer to [2, p. 636–637] and [5]. The non-trivial solution (x, y, z) of Eq. (3) [respectively, Eq. (4)] means that $f(x)f(y) \neq 0$ [respectively $f(x)f(y) \neq 0$ and $f(x) \neq f(y)$].

Using the theory of Pell equation, we prove:

Theorem 2.1. *When $f(x) = \frac{x(Bx+C)}{2}$ with positive integer B and non-zero integer C, Eqs. (3) and (4) have infinitely many non-trivial positive integer solutions.*

Theorem 2.2. *When*

$$f(x) = \prod_{i=0}^{m-1}(x - i)\ (m \geq 2),$$

Eq. (3) has infinitely many non-trivial positive integer solutions.

Theorem 2.3. *When*

$$f(x) = x\prod_{i=0}^{m-2}(x - k^i)\ (m \geq 3, k \geq 2),$$

Eqs. (3) and (4) have infinitely many non-trivial positive integer solutions.

Theorem 2.4. *When*

$$f(x) = x\prod_{i=0}^{m-2}(x - \sum_{j=0}^{i} k^i)\ (m \geq 3,\ k \geq 2),$$

Eqs. (3) and (4) have infinitely many non-trivial positive integer solutions.

In Theorems 2.2, 2.3 and 2.4, when $m = 2$, $f(x) = x(x - 1)$, it is the case $B = 2,\ C = -2$ in Theorem 2.1.

3 Proofs of the Theorems

Proof of Theorem 2.1.

Case 1) The case for Eq. (3).

For $f(x)=\frac{x(Bx+C)}{2}$, put $y=2t(Bx+C)$, then Eq. (3) reduces to

$z^2=\frac{(Bx+C)^2}{4}\left((16B^4t^4+4B^2t^2+1)x^2+2Ct(2Bt+1)(8B^2t^2+1)x+4C^2t^2(2Bt+1)^2\right).$

To get integral values of x and z, it needs to study the integer solutions x and s of the quadratic equation

$$4s^2=(16B^4t^4+4B^2t^2+1)x^2+2Ct(2Bt+1)(8B^2t^2+1)x+4C^2t^2(2Bt+1)^2.$$

Let

$$X=D_1x+M_1,\ Y=2s,$$

where

$$D_1=(4B^2t^2)^2+4B^2t^2+1,\ M_1=Ct(2Bt+1)(8B^2t^2+1),$$

then we get the Pell equation

$$X^2-D_1Y^2=-3C^2t^2(2Bt+1)^2. \tag{5}$$

It is easy to show that $D_1=(4B^2t^2)^2+4B^2t^2+1$ is not a perfect square for $t>2,\ B>0$, then the Pell equation $X^2-D_1Y^2=1$ has infinitely many positive integer solutions. Suppose that (u,v) is a positive integer solution of $X^2-D_1Y^2=1$.

1.1) When $B>0,\ C<0$, note that

$$(X_0,Y_0)=\big(Ct(2Bt+1)(8B^2t^2+1),-2Ct(2Bt+1)\big)$$

is an integer solution of Eq. (5), then an infinity of integer solutions of Eq. (5) are given by

$$\begin{aligned}X_n+Y_n\sqrt{D_1}=&\left(Ct(2Bt+1)(8B^2t^2+1)-2Ct(2Bt+1)\sqrt{D_1}\right)\\&\times\left(u+v\sqrt{D_1}\right)^n,\ n\geq 0,\end{aligned}$$

which leads to

$$X_n=uX_{n-1}+D_1vY_{n-1},\ Y_n=uY_{n-1}+vX_{n-1}.$$

Thus,

$$\begin{cases}X_n=2uX_{n-1}-X_{n-2}, & X_0=Ct(2Bt+1)(8B^2t^2+1),\\ & X_1=-Ct(2Bt+1)(2D_1v-(8B^2t^2+1)u);\\ Y_n=2uY_{n-1}-Y_{n-2}, & Y_0=-2Ct(2Bt+1),\\ & Y_1=Ct(2Bt+1)(8B^2t^2v-2u+v).\end{cases}$$

Using the recurrence relations of X_n and Y_n twice, we get

$$X_{n+2}=(4u^2-2)X_n-X_{n-2},\ Y_{n+2}=(4u^2-2)Y_n-Y_{n-2}.$$

Replacing n by $2n$, we have

$$X_{2n+2}=(4u^2-2)X_{2n}-X_{2n-2},\ Y_{2n+2}=(4u^2-2)Y_{2n}-Y_{2n-2},\ n\geq 1.$$

From

$$x = \frac{X - M_1}{D_1},\ s = \frac{Y}{2},$$

we obtain

$$\begin{cases} x_{2n+2} = (4u^2-2)x_{2n} - x_{2n-2} + 4M_1v^2, & x_0 = 0, \\ & x_2 = 2Ct(2Bt+1)v((8B^2t^2+1)v - 2u); \\ s_{2n+2} = (4u^2-2)s_{2n} - s_{2n-2}, & s_0 = -Ct(2Bt+1), \\ & s_2 = -Ct(2Bt+1)(u^2 - (8B^2t^2+1)vu + v^2D_1). \end{cases}$$

From

$$u^2 = 1 + (16B^4t^4 + 4B^2t^2 + 1)v^2,$$

we have

$$\big((8B^2t^2+1)v\big)^2 - (2u)^2 = -3v^2 - 4 < 0,$$

so

$$(8B^2k^2+1)v - 2u < 0.$$

In view of

$$C < 0,\ v > 0,$$

we have

$$x_2 \in \mathbb{Z}^+,$$

and

$$x_2 > -2C > -\frac{2C}{B}.$$

From the recurrence relation of x_{2n}, it is easy to check that

$$x_{2n} > -2C > -\frac{2C}{B}$$

and

$$2k(Bx_{2n} + C) > x_{2n}$$

hold for $n \geq 1$, which means that $y_{2n} = 2k(Bx_{2n} + C) > x_{2n}$. For x in $[-\frac{2C}{B}, +\infty)$, the polynomial $f(x) = \frac{x(Bx+C)}{2}$ is strictly monotonically increasing and $f(-\frac{2C}{B}) > 0$, so

$$f(y_{2n}) > f(x_{2n}) > 0,\ n \geq 1.$$

Thus, when $f(x) = \frac{x(Bx+C)}{2}$ with positive integer B and negative integer C, Eq. (3) has infinitely many non-trivial positive integer solutions

$$(x, y, z) = (x_{2n},\ 2t(Bx_{2n} + C),\ s_{2n}(Bx_{2n} + C)),$$

where $n \geq 1$.

1.2) When $B > 0,\ C > 0$, note that

$$(X_0, Y_0) = \big(Ct(2Bt+1)(8B^2t^2+1), 2Ct(2Bt+1)\big)$$

is a positive integer solution of Eq. (5).

The remaining process can be given by similar way.

Case 2) The case for Eq. (4).

For $f(x) = \frac{x(Bx+C)}{2}$, take $y = 2t(Bx+C)$, then Eq. (4) becomes

$$z^2 = \frac{(Bx+C)^2}{4}\left((16B^4t^4 - 4B^2t^2+1)x^2 + 2Ct(2Bt+1)(8B^2t^2-1)x + 4C^2t^2(2Bt+1)^2\right).$$

We need to study the equation

$$4s^2 = (16B^4t^4 - 4B^2t^2+1)x^2 + 2Ct(2Bt+1)(8B^2t^2-1)x + 4C^2t^2(2Bt+1)^2.$$

Put

$$X = D_2x + M_2,\ Y = 2s,$$

where

$$D_2 = (4B^2t^2)^2 - 4B^2t^2 + 1,\ M_2 = Ct(2Bt+1)(8B^2t^2-1),$$

we have

$$X^2 - D_2Y^2 = -3C^2t^2(2Bt+1)^2. \tag{6}$$

Obviously, $D_2 = (4B^2t^2)^2 - 4B^2t^2 + 1$ is not a perfect square for $t > 2,\ B > 0$, so the Pell equation $X^2 - D_2Y^2 = 1$ has infinitely many positive integer solutions. Assume that (u, v) solves the Pell equation $X^2 - D_2Y^2 = 1$.

2.1) When $B > 0,\ C < 0$,

$$(X_0, Y_0) = \left(Ct(2Bt+1)(8B^2t^2-1), -2Ct(2Bt+1)\right)$$

is an integer solution of Eq. (6).
Similarly, we can give the remainder of the proof as above.

2.2) When $B > 0,\ C > 0$, Eq. (6) has a positive integer solution

$$(X_0, Y_0) = \left(Ck(2Bt+1)(8B^2t^2-1), 2Ct(2Bt+1)\right).$$

The remainder of the proof is similar as 1.1) above. $\square$

Example 3.1. *When $B = k-2,\ C = -(k-4),\ k > 4,\ f(x) = \frac{x\left((k-2)x-(k-4)\right)}{2}$, which denotes the polygonal number, then Eq. (3) has infinitely many non-trivial positive integer solutions*

$$(x, y, z) = (x_{2n},\ 2t\left((k-2)x_{2n} - (k-4)\right),\ \left((k-2)x_{2n} - (k-4)\right)s_{2n}),$$

where

$$\begin{cases} x_{2n+2} = (4u^2-2)x_{2n} - x_{2n-2} + 4M_1v^2, & x_0 = 0, \\ & x_2 = -2tv(k-4)(2t(k-2)+1) \\ & \qquad (8(k-2)^2t^2v + v - 2u); \\ s_{2n+2} = (4u^2-2)s_{2n} - s_{2n-2}, & s_0 = t(k-4)(2t(k-2)+1), \\ & s_2 = t(k-4)(2t(k-2)+1) \\ & \qquad (u^2 - (8(k-2)^2t^2+1)vu + v^2D_1), \end{cases}$$

$$\begin{aligned}D_1 =&16(k-2)^4t^4+4(k-2)^2t^2+1,\\ M_1 =&-t(k-4)(2t(k-2)+1)(8(k-2)^2t^2+1)\end{aligned}$$

and (u,v) *satisfying*

$$u^2-(16(k-2)^4t^4+4(k-2)^2t^2+1)v^2=1.$$

Remark 3.2. *If there exist two similar integral triangles, as constructed in Theorem 2.1, then there are* $n_1,\ n_2\in\mathbb{Z}^+$ $(n_1\neq n_2)$ *such that*

$$\frac{f(x_{2n_1})}{f(y_{2n_1})}=\frac{f(x_{2n_2})}{f(y_{2n_2})}.$$

The above equality is satisfied if and only if

$$x_{2n_1}=x_{2n_2},$$

which is obviously impossible. Therefore, the integral triangles are not similar in Theorem 2.1.

Proof of Theorem 2.2. For $f(x)=\prod_{i=0}^{m-1}(x-i)$, let $y=x-1$, then Eq. (3) leads to

$$\left(x^2+x(x-m)+(x-m)^2\right)\prod_{i=1}^{m-1}(x-i)^2=z^2.$$

This implies that

$$x^2+x(x-m)+(x-m)^2=s^2,$$

which is equivalent to the Pell equation

$$X^2-3Y^2=-3m^2, \tag{7}$$

where

$$X=6x-3m,\ Y=2s.$$

Let us observe that $(X_0,Y_0)=(3m,2m)$ is a positive integer solution of Eq. (7), and $(u,v)=(2,1)$ is the least positive integer solution of $X^2-3Y^2=1$.
The remainder of the proof can be given by similar method. □

Example 3.3. *When* $m=3$, $f(x)=x(x-1)(x-2)$, *Eq. (3) becomes*

$$x^2(x-1)^2(x-2)^2+x(x-1)(x-2)y(y-1)(y-2)+y^2(y-1)^2(y-2)^2=z^2.$$

It has infinitely many non-trivial positive integer solutions

$$(x,y,z)=(x_{2n},\ x_{2n}-1,\ (x_{2n}-1)(x_{2n}-2)s_{2n}),$$

where

$$\begin{cases}x_{2n+2}=14x_{2n}-x_{2n-2}-18, & x_0=3,\ x_2=24;\\ s_{2n+2}=14s_{2n}-s_{2n-2}, & s_0=3,\ s_2=39,\end{cases}$$

and $n\geq 1$.

Remark 3.4. *For $f(x) = \prod_{i=0}^{m-1}(x-i)$, we study the non-trivial positive integer solutions of Eq. (4) for some m. When $m = 3$, it has infinity many non-trivial positive solutions*

$$(x, y, z) = (x_{2n}, 2x_{2n} - 2, s_{2n}(x_{2n} - 1)(x_{2n} - 2)),$$

where

$$\begin{cases} x_{2n+2} = 91202x_{2n} - x_{2n-2} - 144000, & x_0 = 0,\ x_2 = 480; \\ s_{2n+2} = 91202s_{2n} - s_{2n-2}, & s_0 = 12,\ s_2 = 3612, \end{cases}$$

and $n \geq 1$. When $m = 4$, in the range $3 < x < y < 10000$, we only find three non-trivial positive integer solutions

$$(x, y, z) = (13, 22, 167640), (147, 513, 68227820640),$$
$$(222, 289, 6009373656).$$

And for $m = 5$, in the range $4 < x < y < 10000$, we get three non-trivial positive integer solutions

$$(x, y, z) = (7, 8, 5880), (20, 21, 2209320), (25, 27, 8528400).$$

To Eq. (4), it seems difficult to get the similar result as Eq. (3) for general m.

Proof of Theorem 2.3. For

$$f(x) = x\prod_{i=0}^{m-2}\left(x - k^i\right)\ (m \geq 3,\ k \geq 2),$$

let $y = kx$, then Eqs. (3) and (4) lead to

$$z^2 = \left(x\prod_{i=0}^{m-3}(x - k^i)\right)^2 \left((x - k^{m-2})^2 \pm k^{m-1}(kx - 1)(x - k^{m-2}) + k^{2m-2}(kx - 1)^2\right).$$

Since we are interested in the integer solutions (x, z) of the above equations, we need to investigate

$$(x - k^{m-2})^2 \pm k^{m-1}(kx - 1)(x - k^{m-2}) + k^{2m-2}(kx - 1)^2 = s^2.$$

1) First we consider the case of sign $+$.
   Put

$$X = 2D_3x - M_3,\ Y = 2s,$$

   where

$$D_3 = (k^m)^2 + k^m + 1,\ M_3 = k^{m-2}(2k^{m+1} + k^m + k + 2),$$

   then

$$X^2 - D_3Y^2 = -3k^{2m-2}(k^{m-1} - 1)^2. \tag{8}$$

   Note that

$$(X_0, Y_0) = \left(k^{m-1}(2k^m + 1)(k^{m-1} - 1), 2k^{m-1}(k^{m-1} - 1)\right)$$

   is a positive integer solution of Eq. (8).
   The rest of the procedure are the same as 1.1) in Theorem 2.1.

2) Then we consider the case of sign $-$.
Put

$$X = 2D_4 x - M_4,\ Y = 2s,$$

where

$$D_4 = (k^m)^2 - k^m + 1,\ M_4 = k^{m-2}(2k^{m+1} - k^m - k + 2),$$

then we obtain

$$X^2 - D_4 Y^2 = -3k^{2m-2}(k^{m-1} - 1)^2. \tag{9}$$

It is easy to see that

$$(X_0, Y_0) = \big(k^{m-1}(2k^m - 1)(k^{m-1} - 1), 2k^{m-1}(k^{m-1} - 1)\big)$$

is a positive integer solution of Eq. (9).
Apparently, we can receive the result which identifies with the Theorem 2.3. □

Example 3.5. *When $m = 3$, $f(x) = x(x-1)(x-k)$, put $y = kx$, Eq. (3) equals*

$$(x(x-1))^2 \left((k^6 + k^3 + 1)x^2 - k(2k^4 + k^3 + k + 2)x + k^2(k^2 + k + 1)\right) = z^2.$$

It has infinitely many non-trivial positive integer solutions

$$(x, y, z) = (x_{2n},\ kx_{2n},\ x_{2n}(x_{2n} - 1)s_{2n}),$$

where

$$\begin{cases} x_{2n+2} = (4u^2 - 2)x_{2n} - x_{2n-2} & \\ \qquad - 2v^2 k(2k^4 + k^3 + k + 2), & x_0 = k, \\ & x_2 = k(k(k-1)(k+1)(2k^3+1)v^2 \\ & \qquad + 2ku(k-1)(k+1)v + 1); \\ s_{2n+2} = (4u^2 - 2)s_{2n} - s_{2n-2}, & s_0 = k^2(k^2 - 1), \\ & s_2 = k^2(k^2-1)(2u^2 + (2k^3+1)vu - 1), \end{cases}$$

and (u, v) is a positive integer solution of

$$X^2 - (k^6 + k^3 + 1)Y^2 = 1.$$

Proof of Theorem 2.4. For

$$f(x) = x \prod_{i=0}^{m-2} \left(x - \sum_{j=0}^{m-2} k^j\right) \ (m \geq 3,\ k \geq 2),$$

let $y = kx + 1,\ A = \sum_{j=0}^{m-2} k^j$, then Eqs. (3) and (4) reduce to

$$z^2 = \left(x \prod_{i=0}^{m-3} (x - \sum_{j=0}^{i} k^j)\right)^2 \left((x-A)^2 \pm k^{m-1}(kx+1)(x-A) + k^{2m-2}(kx+1)^2\right).$$

Let us study the equations

$$(x-A)^2 \pm k^{m-1}(kx+1)(x-A) + k^{2m-2}(kx+1)^2 = s^2.$$

1) Look the sign $+$.
   Take
$$X = 2D_5x - M_5,\ Y = 2s,$$
   where
$$D_5 = (k^m)^2 + k^m + 1,\ M_5 = -2k^{2m-1} + Ak^m - k^{m-1} + 2A,$$
   then we get the Pell equation
$$X^2 - D_5Y^2 = -3k^{2m-2}(Ak+1)^2. \tag{10}$$
   We find that the pair
$$(X_0, Y_0) = \big(k^{m-1}(2k^m+1)(Ak+1), 2k^{m-1}(Ak+1)\big)$$
   is a positive integer solution of Eq. (10).
   Following the method of 1.1) in Theorem 2.1, the result is clearly established.

2) When the sign is $-$.
   Let
$$X = 2D_6x + M_6,\ Y = 2s,$$
   where
$$D_6 = (k^m)^2 - k^m + 1,\ M_6 = 2k^{2m-1} + Ak^m - k^{m-1} - 2A,$$
   then we get the Pell equation
$$X^2 - D_6Y^2 = -3k^{2m-2}(Ak+1)^2. \tag{11}$$
   Note that
$$(X_0, Y_0) = \big(k^{m-1}(2k^m-1)(Ak+1), 2k^{m-1}(Ak+1)\big)$$
   is a positive integer solution of Eq. (11).
   The remainder of the proof is similar as 1.1). □

Example 3.6. *When $m = 3$, $f(x) = x(x-1)(x-1-k)$, put $y = kx+1$, Eq. (3) equals*
$$(x(x-1))^2\left((k^6+k^3+1)x^2 + (2k^5 - Ak^3 + k^2 - 2A)x + k^4 - Ak^2 + A^2\right) = z^2,$$
where $A = 1+k$. It has infinitely many non-trivial positive integer solutions
$$(x, y, z) = (x_{2n}, kx_{2n}+1, x_{2n}(x_{2n}-1)s_{2n}),$$
where
$$\begin{cases} x_{2n+2} = (4u^2-2)x_{2n} - x_{2n-2} & \\ \qquad\quad - 2v^2(-2k^5 + Ak^3 - k^2 + 2A), & x_0 = A, \\ & x_2 = k^2v(2vk^3 + 2u + v)(Ak+1) + A; \\ s_{2n+2} = (4u^2-2)s_{2n} - s_{2n-2}, & s_0 = k^2(Ak+1), \\ & s_2 = k^2(2u^2 + (2k^3+1)vu - 1)(Ak+1), \end{cases}$$
and (u, v) is a positive integer solution of
$$X^2 - (k^6+k^3+1)Y^2 = 1.$$

4 Some related questions

In our theorems the polynomials $f(x)$ are reducible. For irreducible polynomials, we did not get similar results. So we raise the following questions.

Question 4.1. *Are there irreducible polynomials $f(x) \in \mathbb{Q}(x)$ with $\deg f(x) \geq 2$ such that Eqs. (3) and (4) have infinitely many non-trivial positive integer solutions?*

For $f(x) = \prod_{i=0}^{m-1}(x-i)$ $(m \geq 2)$, by searching on computer, we find Eq. (4) has some non-trivial positive integer solutions for $m = 4,\ 5$ (see Remark 3.4), but fail to obtain infinitely many ones. Therefore, we have:

Question 4.2. *Does Eq. (4) have infinitely many non-trivial positive integer solutions for*

$$f(x) = \prod_{i=0}^{m-1}(x-i),$$

where $m \geq 4$?

Noting that the areas of the integral triangles, constructed in our Theorems, are

$$A = \frac{f(x)f(y)\sin(\theta)}{2},$$

which are not rational, so they are not Heron triangles. It is natural to ask:

Question 4.3. *Are there Heron triangles whose two adjacent sides are given by the values of polynomials $f(x)$ and $f(y)$ with a fixed Heron angle?*

For any angle θ, if $\sin(\theta)$ and $\cos(\theta)$ are rational, then we call θ is a Heron angle. In other words, Question 4.3 is equivalent to the existence of positive integer solutions (x, y, z) to the Diophantine system

$$\begin{cases} z^2 = f(x)^2 - 2f(x)f(y)\cos(\theta) + f(y)^2, \\ A = \dfrac{f(x)f(y)\sin(\theta)}{2} \in \mathbb{Z}^+, \end{cases} \tag{12}$$

where $\cos(\theta) = \dfrac{1-s^2}{1+s^2}$ and $\sin(\theta) = \dfrac{2s}{1+s^2}$.

When $s = 1/2$, we have $\cos(\theta) = \dfrac{3}{5}$ and $\theta = 37°$. For $f(x) = x(x+1)$, we can use the same method in Theorem 2.1 to show that $z^2 = f(x)^2 - 2f(x)f(y)\cos(37°) + f(y)^2$ has infinitely many non-trivial positive integer solutions

$$(x, y, z) = (x_n, x_n + 1, 2(x_n + 1)s_n),\ n \geq 1,$$

where

$$\begin{cases} x_n = 18x_{n-1} - x_{n-2} + 16, & x_0 = 0,\ x_1 = 28; \\ s_n = 18s_{n-1} - s_{n-2}, & s_0 = 1,\ s_1 = 13. \end{cases}$$

From the recurrence relation of x_n, it is easy to check that

$$x_{2n} \equiv 0 \pmod 5,\ x_{2n-1} \equiv 3 \pmod 5,\ n \geq 1,$$

which means that

$$x_n(x_n+1)^2(x_n+2) \equiv 0 \pmod 5$$

holds for $n \geq 1$. Therefore,

$$A_n = \frac{f(x_n)f(y_n)\sin(37^\circ)}{2} = \frac{2x_n(x_n+1)^2(x_n+2)}{5} \in \mathbb{Z}^+.$$

So Eq. (12) has infinitely many non-trivial positive integer solutions.

In 2017, Sz. Tengely and M. Ulas [9] showed that the Diophantine equations

$$z^2 = f(x)^2 \pm g(y)^2$$

have infinitely many non-trivial polynomials solutions with integer coefficients for $f(x) = x^k(x+a),\ g(x) = x^k(x+b)$ with $k \geq 1,\ a^2+b^2 \neq 0$. Similarly, we can raise:

Question 4.4. *Are there polynomials $f(x),\ g(y) \in \mathbb{Q}(x)$ such that the Diophantine equations*

$$z^2 = f(x)^2 + f(x)g(y) + g(y)^2$$

and

$$z^2 = f(x)^2 - f(x)g(y) + g(y)^2$$

have infinitely many non-trivial positive integer solutions?

It seems that there exist some interesting results for Questions 4.3 and 4.4, we hope to come back to study them in the near future.

References

[1] Burn, B. (2003). Triangles with a 60° angle and sides of integer length. *The Mathematical Gazette*, 87(508), 148–153.

[2] Dickson, L. E. (2005). *History of the Theory of Numbers, Vol. II: Diophantine Analysis*. Dover Publications.

[3] Gilder, J. (1982). Integer-sided triangles with an angle of 60°. *The Mathematical Gazette*, 66(438), 261–266.

[4] He, B., Togbé, A., & Ulas, M. (2010). On the Diophantine equation $z^2 = f(x)^2 \pm f(y)^2$, II. *Bulletin of the Australian Mathematical Society*, 82(2), 187–204.

[5] Pocklington, H. C. (1914). Some Diophantine impossibilities. *Mathematical Proceedings of the Cambridge Philosophical Society*, 17, 110–118.

[6] Read, E. (2006). On integer-sided triangles containing angles of 120° or 60°. *The Mathematical Gazette*, 90(518), 299–305.

[7] Selkirk, K. (1983). Integer-sided triangles with an angle of $120°$. *The Mathematical Gazette*, 67(442), 251–255.

[8] Sierpiński, W. (1962). *Triangular Numbers*. Biblioteczka Matematyczna 12, Warszawa (in Polish).

[9] Tengely, Sz., & Ulas, M. (2017). On certain Diophantine equations of the form $z^2 = f(x)^2 \pm g(y)^2$. *Journal of Number Theory*, 174, 239–257.

[10] Ulas, M., & Togbé, A. (2010). On the Diophantine equation $z^2 = f(x)^2 \pm f(y)^2$. *Publicationes Mathematicae Debrecen*, 76(1–2), 183–201.

[11] Youmbai, A. E. A., & Behloul, D. (2019). Rational solutions of the Diophantine equations $f(x)^2 \pm f(y)^2 = z^2$. *Periodica Mathematica Hungarica*, 79(2), 255–260.

[12] Zhang, Y., & Tang, Q. Z. (2021). On the integer solutions of the Diophantine equations $z^2 = f(x)^2 \pm f(y)^2$. *Periodica Mathematica Hungarica*, accepted.

[13] Zhang, Y., Tang, Q. Z., & Zhang, Y. N. (2020). On the Diophantine equations $z^2 = f(x)^2 \pm f(y)^2$ involving Laurent polynomials, II. submitted.

[14] Zhang, Y., & Zargar, A. S. (2019). On the Diophantine equations $z^2 = f(x)^2 \pm f(y)^2$ involving quartic polynomials. *Periodica Mathematica Hungarica*, 79(1), 25–31.

[15] Zhang, Y., & Zargar, A. S. (2020). On the Diophantine equations $z^2 = f(x)^2 \pm f(y)^2$ involving Laurent polynomials. *Functiones et Approximatio, Commentarii Mathematici*, 62(2), 187–201.