Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 27, 2021, No. 2, 79–87
DOI: 10.7546/nntdm.2021.27.2.79-87

Regular polygons, Morgan-Voyce polynomials, and Chebyshev polynomials

Jorma K. Merikoski

Faculty of Information Technology and Communication Sciences, Tampere University
FI-33014 Tampere, Finland
e-mail: jorma.merikoski@tuni.fi

Received: 14 July 2020 **Revised:** 9 April 2021 **Accepted:** 18 April 2021

Abstract: We say that a monic polynomial with integer coefficients is a polygomial if its each zero is obtained by squaring the edge or a diagonal of a regular n-gon with unit circumradius. We find connections of certain polygomials with Morgan-Voyce polynomials and further with Chebyshev polynomials of second kind.
Keywords: Regular polygons, Morgan-Voyce polynomials, Chebyshev polynomials, Vieta polynomials.
2020 Mathematics Subject Classification: 11B83, 51M20.

1 Introduction

We call the edge and diagonals of a polygon by a common name *chord*. Let G_n be a regular n-gon with unit circumradius. Its chords (their lengths) are

$$e_{nk} = 2\sin\frac{k\pi}{n}, \quad k = 1, \dots, \left\lfloor\frac{n}{2}\right\rfloor.$$

We say that a monic polynomial with integer coefficients is a *polygomial* if its all zeros are squared chords (not necessarily squares of all chords) of some G_n.

This paper is a sequel to Mustonen et al. [7, Section 2] on the polygomials

$$A_m(x) = \sum_{k=0}^{m}(-1)^{m-k}\binom{m+k+1}{2k+1}x^k = \prod_{k=1}^{m}\left[x - 4\sin^2\frac{k\pi}{2(m+1)}\right] \tag{1}$$

(for the second equation, see [7, Theorem 1]) and

$$\tilde{A}_m(x) = x^m + \sum_{k=0}^{m-1} (-1)^{m-k} \frac{2m+1}{m-k} \binom{m+k}{2k+1} x^k = \prod_{k=1}^{m} \left[x - 4\sin^2 \frac{k\pi}{2m+1} \right]$$

(for the second equation, see [7, Theorem 2]). Define that the "empty sum" is zero and the "empty product" one; then $A_0(x) = \tilde{A}_0(x) = 1$. In [7], $B_m = \tilde{A}_m$ and $\tilde{A}_m$ has another meaning.

Example 1.1. In particular,

$$A_1(x) = x - 2, \quad A_2(x) = x^2 - 4x + 3, \quad A_3(x) = x^3 - 6x^2 + 10x - 4,$$
$$A_4(x) = x^4 - 8x^3 + 21x^2 - 20x + 5,$$
$$\tilde{A}_1(x) = x - 3, \quad \tilde{A}_2(x) = x^2 - 5x + 5, \quad \tilde{A}_3(x) = x^3 - 7x^2 + 14x - 7,$$
$$\tilde{A}_4(x) = x^4 - 9x^3 + 27x^2 - 30x + 9.$$

The sequence (A_m) satisfies [7, Equation (6)] the recursion

$$A_0(x) = 1, \quad A_1(x) = x - 2, \quad A_{m+1}(x) = (x-2)A_m(x) - A_{m-1}(x), \tag{2}$$

and $(\tilde{A}_m)$ satisfies [7, Equation (11)]

$$\tilde{A}_0(x) = 1, \quad \tilde{A}_1(x) = x - 3, \quad \tilde{A}_{m+1}(x) = (x-3)A_m(x) - A_{m-1}(x). \tag{3}$$

We show that

$$\tilde{A}_0(x) = 1, \quad \tilde{A}_1(x) = x - 3, \quad \tilde{A}_{m+1}(x) = (x-2)\tilde{A}_m(x) - \tilde{A}_{m-1}(x). \tag{4}$$

Thus $(\tilde{A}_m)$ follows the same recursion formula as (A_m).

For all $k \geq 2$,

$$\begin{aligned}\tilde{A}_k(x) &\overset{(3)}{=} (x-3)A_{k-1}(x) - A_{k-2}(x) \\ &\overset{(2)}{=} (x-3)A_{k-1}(x) + A_k(x) - (x-2)A_{k-1}(x) = A_k(x) - A_{k-1}(x).\end{aligned} \tag{5}$$

Therefore

$$\begin{aligned}&\tilde{A}_{m+1}(x) - (x-2)\tilde{A}_m(x) + \tilde{A}_{m-1}(x) \\ &\overset{(5)}{=} A_{m+1}(x) - A_m(x) - (x-2)(A_m(x) - A_{m-1}(x)) + A_{m-1}(x) - A_{m-2}(x) \\ &= A_{m+1}(x) - (x-2)A_m(x) + A_{m-1}(x) - [A_m(x) - (x-2)A_{m-1}(x) + A_{m-2}(x)] \\ &\overset{(2)}{=} 0 - 0 = 0,\end{aligned}$$

verifying the claim.

We are interested in connections of A_m and $\tilde{A}_m$ with well-known polynomials. We introduce in Sections 3 and 4 the Morgan-Voyce polynomials b_m and B_m, and their generalizations $B_m^{(r)}$. We see that A_m and $\tilde{A}_m$ are connected with B_m and $B_m^{(2)}$, respectively. We also find a polygomial a_m connected with b_m. In Section 5, recalling how b_m, B_m, and $B_m^{(2)}$ reduce to the Chebyshev polynomials of second kind, we reduce also a_m, A_m, and $\tilde{A}_m$ to them. The motivation of b_m and B_m rises from a problem on a ladder network of resistances. We see in Section 6 that also a_m and A_m apply to this problem. Finally, we complete this paper with conclusions and remarks in Section 7.

2 Background

Let me first describe the background of this paper. Neeme Vaino, an amateur mathematician from Estonia, introduced [11] his “regular polynomials”

$$R_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} r_{nk} x^{n-2k},$$

whose coefficients are obtained from the OEIS [8] sequence A132460. Actually [5, Equation (1.5)] R_n is the *Vieta–Lucas polynomial*

$$v_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} x^{n-2k}.$$

Example 2.1. In particular,

$$v_1(x) = x, \quad v_2(x) = x^2 - 2, \quad v_3(x) = x^3 - 3x, \quad v_4(x) = x^4 - 4x^2 + 2,$$
$$v_5(x) = x^5 - 5x^3 + 5x, \quad v_6(x) = x^6 - 6x^4 + 9x^2 - 2.$$

The polygomial $\tilde{A}_m$ relates to v_{2m+1} via

$$v_{2m+1}(x) = x\tilde{A}_m(x^2),$$

cf. [5, Theorem 3(b)].

Example 2.2. In particular,

$$x\tilde{A}_2(x^2) = x(x^4 - 5x^2 + 5) = x^5 - 5x^3 + 5x = v_5(x).$$

The *Chebyshev polynomials of first kind* are defined by

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

Example 2.3. In particular,

$$T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x, \quad T_4(x) = 8x^4 - 8x^2 + 1,$$
$$T_5(x) = 16x^5 - 20x^3 + 5x, \quad T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1.$$

The polynomial v_n relates to T_n [5, Equation (9.4)] via $v_n(x) = 2T_n(\frac{x}{2})$.

Example 2.4. In particular,

$$2T_4(\frac{x}{2}) = 2 \cdot \left[8\left(\frac{x}{2}\right)^4 - 8\left(\frac{x}{2}\right)^2 + 1\right] = x^4 - 4x^2 + 2 = v_4(x).$$

The *Vieta–Fibonacci polynomials* are defined by

$$V_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} x^{n-2k},$$

cf. [5, Equation (1.3)]. Here V_n denotes the same as V_{n+1} in [5], in order to make its degree equal to n.

Example 2.5. In particular,

$$V_1(x) = x, \quad V_2(x) = x^2 - 1, \quad V_3(x) = x^3 - 2x, \quad V_4(x) = x^4 - 3x^2 + 1,$$
$$V_5(x) = x^5 - 4x^3 + 3x, \quad V_6(x) = x^6 - 5x^4 + 6x^2 - 1.$$

The polygomial A_m relates to V_{2m+1} via $V_{2m+1}(x) = xA_m(x^2)$, cf. [5, Theorem 2(a)].

Example 2.6. In particular,

$$xA_2(x^2) = x(x^4 - 4x^2 + 3) = x^5 - 4x^3 + 3x = V_5(x).$$

The *Chebyshev polynomials of second kind* are defined by

$$U_0(x) = 1, \quad U_1(x) = 2x, \quad U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x).$$

Example 2.7. In particular,

$$U_1(x) = 2x, \quad U_2(x) = 4x^2 - 1, \quad U_3(x) = 8x^3 - 4x, \quad U_4(x) = 16x^4 - 12x^2 + 1,$$
$$U_5(x) = 32x^5 - 32x^3 + 6x, \quad U_6(x) = 64x^6 - 80x^4 + 24x^2 - 1.$$

The polynomial V_n relates to U_n via $V_n(x) = U_n(\frac{x}{2})$, cf. [5, Equation (9.3)]. Actually $V_n(x)$ should read $V_{n+1}(x)$ in this reference, in order to make (9.3) compatible with (1.3).

Example 2.8. In particular,

$$U_4\left(\frac{x}{2}\right) = 16\left(\frac{x}{2}\right)^4 - 12\left(\frac{x}{2}\right)^2 + 1 = x^4 - 3x^2 + 1 = V_4(x).$$

To summarize, A_m relates to V_{2m+1} and further to U_{2m+1}, and $\tilde{A}_m$ relates to v_{2m+1} and further to T_{2m+1}. But we will see that A_m and $\tilde{A}_m$ have also more direct relations to well-known polynomials.

3 Morgan-Voyce polynomials

Changing in A_m all minus signs into plus, we define

$$B_m(x) = (-1)^m A_m(-x). \tag{6}$$

This is one of the two *Morgan-Voyce polynomials* b_m and B_m, usually defined by the recursion pair

$$b_0(x) = B_0(x) = 1, \quad b_{m+1}(x) = xB_m(x) + b_m(x), \quad B_{m+1}(x) = (x+1)B_m(x) + b_m(x). \tag{7}$$

Example 3.1. In particular,

$$b_1(x) = x + 1, \quad b_2(x) = x^2 + 3x + 1, \quad b_3(x) = x^3 + 5x^2 + 6x + 1,$$
$$b_4(x) = x^4 + 7x^3 + 15x^2 + 10x + 1,$$
$$B_1(x) = x + 2, \quad B_2(x) = x^2 + 4x + 3, \quad B_3(x) = x^3 + 6x^2 + 10x + 4,$$
$$B_4(x) = x^4 + 8x^3 + 21x^2 + 20x + 5.$$

By (6) and (2),

$$B_0(x) = 1, \quad B_1(x) = x + 2, \quad B_{m+1}(x) = (x+2)B_m(x) - B_{m-1}(x).$$

This recursion is well-known [9, p. 73] as a consequence of (7). Likewise,

$$b_0(x) = 1, \quad b_1(x) = x + 1, \quad b_{m+1}(x) = (x+2)b_m(x) - b_{m-1}(x).$$

By (6) and (1),

$$B_m(x) = \sum_{k=0}^{m} \binom{m+k+1}{2k+1} x^k = \prod_{k=1}^{m} \left[x + 4\sin^2 \frac{k\pi}{2(m+1)} \right].$$

Regarding zeros, this is well-known [9, Equation (39)] (the first of the two equations with this number) and [10, Section 6]. Similarly, by [9, Equation (40)] and [10, Section 6] (containing a typo),

$$b_m(x) = \sum_{k=0}^{m} \binom{m+k}{2k} x^k = \prod_{k=1}^{m} \left[x + 4\sin^2 \frac{(2k-1)\pi}{2(2m+1)} \right]. \tag{8}$$

4 Counterparts of $\tilde{A}_m$ and b_m

Equation (6) connects A_m and B_m but does not connect $\tilde{A}_m$ and b_m. Instead, b_m is connected with

$$\begin{aligned} a_m(x) = (-1)^m b_m(-x) &\overset{(8)}{=} \prod_{k=1}^{m} (-1) \Big[-x + 4\sin^2 \frac{(2k-1)\pi}{2(2m+1)} \Big] \\ &= \prod_{k=1}^{m} \Big[x - 4\sin^2 \frac{(2k-1)\pi}{2(2m+1)} \Big]. \end{aligned} \tag{9}$$

This is a polygomial, since its zeros are $e^2_{4m+2,1}, e^2_{4m+2,3}, \ldots, e^2_{4m+2,2m-1}$.

Example 4.1. In particular,

$$a_1(x) = x - 1, \quad a_2(x) = x^2 - 3x + 1, \quad a_3(x) = x^3 - 5x^2 + 6x - 1,$$
$$a_4(x) = x^4 - 7x^3 + 15x^2 - 10x + 1.$$

The Morgan-Voyce polynomials have been widely generalized, see [4] and its references. André-Jeannin [1] generalizes them by the recursion

$$B_0^{(r)}(x) = 1, \quad B_1^{(r)}(x) = x + r + 1, \quad B_{m+1}^{(r)}(x) = (x+2)B_m^{(r)}(x) - B_{m-1}^{(r)}(x),$$

where r is a given real number. In particular,

$$B_m^{(0)} = b_m, \quad B_m^{(1)} = B_m,$$

and the polynomials

$$\tilde{B}_m = B_m^{(2)}$$

satisfy the recursion

$$\tilde{B}_0(x) = 1, \quad \tilde{B}_1(x) = x + 3, \quad \tilde{B}_{m+1}(x) = (x+2)\tilde{B}_m(x) - \tilde{B}_{m-1}(x). \tag{10}$$

By (4) and (10), it is easy to see that

$$\tilde{B}_m(x) = (-1)^m \tilde{A}_m(-x). \tag{11}$$

Example 4.2. In particular,

$$\tilde{B}_1(x) = x + 3, \quad \tilde{B}_2(x) = x^2 + 5x + 5, \quad \tilde{B}_3(x) = x^3 + 7x^2 + 14x + 7,$$
$$\tilde{B}_4(x) = x^4 + 9x^3 + 27x^2 + 30x + 9.$$

5 Chebyshev polynomials of second kind

It can be shown [4, Equations (4.2–4)] that

$$B_m(x) = U_m\left(\frac{x+2}{2}\right), \tag{12}$$

$$b_m(x) = U_m\left(\frac{x+2}{2}\right) - U_{m-1}\left(\frac{x+2}{2}\right), \tag{13}$$

$$\tilde{B}_m(x) = U_m\left(\frac{x+2}{2}\right) + U_{m-1}\left(\frac{x+2}{2}\right). \tag{14}$$

Example 5.1. In particular,

$$U_1\left(\frac{x+2}{2}\right) = 2\,\frac{x+2}{2} = x + 2 = B_1(x),$$
$$U_2\left(\frac{x+2}{2}\right) = 4\left(\frac{x+2}{2}\right)^2 - 1 = x^2 + 4x + 3 = B_2(x),$$
$$U_2\left(\frac{x+2}{2}\right) - U_1\left(\frac{x+2}{2}\right) = x^2 + 3x + 1 = b_2(x),$$
$$U_2\left(\frac{x+2}{2}\right) + U_1\left(\frac{x+2}{2}\right) = x^2 + 5x + 5 = \tilde{B}_2(x).$$

It is easy to see that

$$U_m(-x) = (-1)^m U_m(x). \tag{15}$$

Now,

$$A_m(x) \overset{(6)}{=} (-1)^m B_m(-x) \overset{(12)}{=} (-1)^m U_m\left(\frac{-x+2}{2}\right) \overset{(15)}{=} (-1)^{2m} U_m\left(\frac{x-2}{2}\right) = U_m\left(\frac{x-2}{2}\right),$$

$$a_m(x) \overset{(9)}{=} (-1)^m b_m(-x) \overset{(13)}{=} (-1)^m \left(U_m\left(\frac{-x+2}{2}\right) - U_{m-1}\left(\frac{-x+2}{2}\right)\right)$$
$$\overset{(15)}{=} (-1)^m \left[(-1)^m U_m\left(\frac{x-2}{2}\right) - (-1)^{m-1} U_{m-1}\left(\frac{x-2}{2}\right)\right] = U_m\left(\frac{x-2}{2}\right) + U_{m-1}\left(\frac{x-2}{2}\right)$$

and

$$\tilde{A}_m(x) \overset{(11)}{=} (-1)^m \tilde{B}_m(-x) \overset{(14)}{=} (-1)^m \left(U_m\left(\frac{-x+2}{2}\right) + U_{m-1}\left(\frac{-x+2}{2}\right)\right)$$
$$\overset{(15)}{=} (-1)^m \left[(-1)^m U_m\left(\frac{x-2}{2}\right) + (-1)^{m-1} U_{m-1}\left(\frac{x-2}{2}\right)\right] = U_m\left(\frac{x-2}{2}\right) - U_{m-1}\left(\frac{x-2}{2}\right).$$

Example 5.2. In particular,

$$U_1\left(\frac{x-2}{2}\right) = 2\,\frac{x-2}{2} = x-2 = A_1(x),$$

$$U_2\left(\frac{x-2}{2}\right) = 4\left(\frac{x-2}{2}\right)^2 - 1 = x^2-4x+3 = A_2(x),$$

$$U_2\left(\frac{x-2}{2}\right) + U_1\left(\frac{x-2}{2}\right) = x^2-3x+1 = a_2(x),$$

$$U_2\left(\frac{x-2}{2}\right) - U_1\left(\frac{x-2}{2}\right) = x^2-5x+5 = \tilde{A}_2(x).$$

6 Revisiting the ladder network

We show that a_m and A_m apply to the same ladder network problem [2, 6] as b_m and B_m. To begin, we present a recursion pair for a_m and A_m. Since

$$\begin{aligned} a_{m+1}(x) &\overset{(9)}{=} (-1)^{m+1} b_{m+1}(-x) \overset{(7)}{=} (-1)^{m+1}\big[(-x)B_m(-x) + b_m(-x)\big] \\ &= x(-1)^m B_m(-x) - (-1)^m b_m(-x) \overset{(6),(9)}{=} xA_m(x) - a_m(x) \end{aligned}$$

and

$$\begin{aligned} A_{m+1}(x) &\overset{(6)}{=} (-1)^{m+1} B_{m+1}(-x) \overset{(7)}{=} (-1)^{m+1}\big[(-x+1)B_m(-x) + b_m(-x)\big] \\ &= (x-1)(-1)^m B_m(-x) - (-1)^m b_m(-x) \overset{(6),(9)}{=} (x-1)A_m(x) - a_m(x), \end{aligned}$$

we have

$$a_0(x) = A_0(x) = 1, \quad a_{m+1}(x) = xA_m(x) - a_m(x), \quad A_{m+1}(x) = (x-1)A_m(x) - a_m(x). \tag{16}$$

We use the figures and notations of Hoggatt and Bicknell [2, Section 1]. Instead of x, we let $-x$ denote the resistance of each component in the upper sidepiece of the ladder. It is reasonable to require that $-x > 0$, i.e., $x < 0$. However, it is not complete nonsense to accept also nonpositive resistances, because we may think that the voltage across these components can be increased externally. Anyway, whether or not to accept nonpositive resistances, it does not effect on the following calculations.

We proceed as in [2, p. 148] but write $R(x) = R$ and $Z_m(x) = Z_n$. Then

$$R(x) = Z_m(x) - x,$$

$$\frac{1}{Z_{m+1}(x)} = \frac{1}{Z_m(x) - x} + 1 = \frac{Z_m(x) - x + 1}{Z_m(x) - x},$$

and

$$Z_{m+1}(x) = \frac{Z_m(x) - x}{Z_m(x) - x + 1}. \tag{17}$$

We show that

$$Z_m(x) = \frac{a_m(x)}{A_m(x)} \tag{18}$$

satisfies (17). Since

$$Z_{m+1}(x) \overset{(18)}{=} \frac{a_{m+1}(x)}{A_{m+1}(x)} \overset{(16)}{=} \frac{xA_m(x) - a_m(x)}{(x-1)A_m(x) - a_m(x)}$$

$$= \frac{a_m(x) - xA_m(x)}{a_m(x) - (x-1)A_m(x)} = \frac{\frac{a_m(x)}{A_m(x)} - x}{\frac{a_m(x)}{A_m(x)} - x + 1} \overset{(18)}{=} \frac{Z_m(x) - x}{Z_m(x) - x + 1},$$

the claim follows.

7 Conclusions and remarks

The polygomial A_m is connected with B_m via the equation (6). The Morgan-Voyce polynomial b_m defines by (9) the polygomial a_m. The polygomial $\tilde{A}_m$ has the connection (11) with the generalized Morgan-Voyce polynomial $\tilde{B}_m = B_m^{(2)}$. Since these Morgan-Voyce polynomials reduce to Chebyshev polynomials of second kind via (12), (13), and (14), also the above-mentioned polygomials reduce to them.

More generally, we define

$$A_m^{(r)}(x) = (-1)^m B^{(r)}(-x).$$

In particular,

$$A_m^{(0)} = a_m, \quad A_m^{(1)} = A_m, \quad A_m^{(2)} = \tilde{A}_m.$$

If $r \in \{0, 1, 2\}$, then $A_m^{(r)}$ is a polygomial. Is it a polygomial also for some other appropriate values of r? To answer, we should find the zeros of $B_m^{(r)}$. I did not find them from the literature. According to Horadam [3, p. 348], André-Jeannin [1] has given them, but actually he [1, p. 231] considered only the cases $r = 0, 1, 2$.

Acknowledgements

Neeme Vaino's "regular polynomials" (discussed in Section 2) motivated this paper. Pentti Haukkanen gave useful references. I thank both of them.

References

[1] André-Jeannin, R. (1994). A generalization of Morgan-Voyce polynomials. *The Fibonacci Quarterly*, 32, 228–231.

[2] Hoggatt, V. E., & Bicknell, M. (1974). A primer for the Fibonacci numbers: Part XIV. *The Fibonacci Quarterly*, 12, 147–156.

[3] Horadam, A. F. (1996). Polynomials associated with generalized Morgan-Voyce polynomials. *The Fibonacci Quarterly*, 34, 342–348.

[4] Horadam, A. F. (1997). A composite of Morgan-Voyce generalizations. *The Fibonacci Quarterly*, 35, 233–239.

[5] Horadam, A, F. (2002). Vieta polynomials. *The Fibonacci Quarterly*, 40, 223–232.

[6] Morgan-Voyce, A. M. (1959). Ladder networks analysis using Fibonacci numbers. *IRE Transactions on Circuit Theory*, 6, 321–322.

[7] Mustonen, S., Haukkanen, P., & Merikoski, J. (2014). Some polynomials associated with regular polygons. *Acta Universitatis Sapientiae, Mathematica*, 6, 178–193.

[8] *OEIS: The On-Line Encyclopedia of Integer Sequences.* Available online at `http://oeis.org/`.

[9] Swamy, M. N. S. (1966). Properties of the polynomials defined by Morgan-Voyce. *The Fibonacci Quarterly*, 4, 73–81.

[10] Swamy, M. N. S. (1968). Further properties of Morgan-Voyce polynomials. *The Fibonacci Quarterly*, 6, 167–175.

[11] Vaino, N. (2020). Private communication.