

Perrin's bivariate and complex polynomials

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Abstract: In this article, a study is carried out around the Perrin sequence, these numbers marked by their applicability and similarity with Padovan's numbers. With that, we will present the recurrence for Perrin's polynomials and also the definition of Perrin's complex bivariate polynomials. From this, the recurrence of these numbers, their generating function, generating matrix and Binet formula are defined.

Keywords: Perrin complex bivariate polynomials, Perrin polynomials, Perrin sequence.

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1 Introduction

Historically, [8] reports that the Perrin's sequence was mentioned implicitly by Edouard Lucas in 1876, however, only in 1899 this sequence was defined by François Perrin. Perrin's sequence is similar to Padovan's sequence, differing only in the initial conditions, this sequence is linear and recurring of integers numbers, denoted by Pe_n . Thus, we have the recurrence defined below [3].

Definition 1. *The recurrence of Perrin's sequence is given by [11]:*

$$Pe_n = Pe_{n-2} + Pe_{n-3}, n \geq 3,$$

having $Pe_0 = 3, Pe_1 = 0$ and $Pe_2 = 2$ as initial conditions.

In order to explore and study Perrin's numbers, Perrin's polynomials and Perrin's complex bivariate polynomials will be studied based on [2], in which the authors define the Fibonacci and Lucas complex bivariate with definitions and theorems inherent to these numbers, where according to Li and MacHenry [7] the Perrin sequences is F -representable. Historically, [1] reports that Fibonacci polynomials were studied for the first time in 1883, by Eugene Charles Catalan (1814–1894) and later by the mathematician Ernest Erich Jacobsthal (1881–1965).

With that, Catalan introduced the Fibonacci family of polynomial functions [4, 10]. Already Kaygisiz and Sahin [6] pointed out that Pell numbers, Pell–Lucas numbers, bivariate Fibonacci numbers, Perrin sequences, and Exponential Perrin sequences have permanent and determinantal representations.

According to [9], the basic bivariate polynomials present an evolutionary process of the polynomial terms of the sequence. That is, first, polynomials are considered with one variable and two variables, then the imaginary component i is inserted, then these polynomials are explored in their complex form.

2 Perrin's polynomials

In Kaygisiz and Sahin [6] defined generalized Perrin polynomials based on the generalized Lucas polynomials, that are a general form of several polynomials and number sequences. Based on the Fibonacci polynomials and in the work of Kaygisiz and Sahin [6], a recurrence for Perrin's polynomials is studied, to further carry out a study on the Perrin's complex bivariate polynomials.

Definition 2. *Perrin's polynomials $p_n(x)$, for $n > 3$, are given by the formula [5]:*

$$p_n(x) = x^2 p_{n-2}(x) + p_{n-3}(x),$$

on what $p_1(x) = 0, p_1(x) = 2, p_2(x) = 3$ and $p_n(1) = P_n$, being the n -th term in Perrin's sequence.

Thus, we have the first terms of this sequence presented in the following Table 1.

n	$p_n(x)$
1	0
2	2
3	3
4	$2x^2$
5	$2 + 3x^2$
6	$3 + 2x^4$
7	$4x^2 + 3x^4$
\vdots	\vdots

Table 1. First terms of Perrin's polynomial sequence

With this, some properties, inherent to this polynomial sequence, are discussed, thus approaching its matrix form, generating function and Binet formula, being, therefore, new ways to obtain the terms of this sequence.

Theorem 2.1. *The generator matrix of Perrin's bivariate polynomials, for $n \geq 1$ and with $n \in \mathbb{N}$, is given by:*

$$vQ^n = \begin{bmatrix} 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ x^2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^n = \begin{bmatrix} p_{n+3}(x) & p_{n+2}(x) & p_{n+1}(x) \end{bmatrix}$$

on what v represents the vector with the sequence initialization values, and Q represents the generating matrix.

Proof. Through the principle of finite induction, we have the following. For $n = 1$, we have:

$$vQ^1 = \begin{bmatrix} 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ x^2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2x^2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} p_4(x) & p_3(x) & p_2(x) \end{bmatrix}.$$

We validate the equality discussed above. Assuming it is valid for $n = k, k \in \mathbb{N}$, we have that:

$$vQ^k = \begin{bmatrix} 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ x^2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^k = \begin{bmatrix} p_{k+3}(x) & p_{k+2}(x) & p_{k+1}(x) \end{bmatrix}$$

Now, verifying that it is valid for $n = k + 1$, we have that:

$$\begin{aligned} vQ^{k+1} &= \begin{bmatrix} 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ x^2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^k \begin{bmatrix} 0 & 1 & 0 \\ x^2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} p_{k+3}(x) & p_{k+2}(x) & p_{k+1}(x) \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ x^2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} x^2 p_{k+2}(x) + p_{k+1}(x) & p_{k+3}(x) & p_{k+2}(x) \end{bmatrix} \\ &= \begin{bmatrix} p_{k+4}(x) & p_{k+3}(x) & p_{k+2}(x) \end{bmatrix}. \end{aligned}$$

□

The characteristic equation of this Perrin polynomial sequence is given by $t^3 - x^2t - 1 = 0$, where x^2 is the polynomial variable. So, we have that t_1, t_2 and t_3 are the roots of the characteristic equation, with: $t_1 = A + B, t_2 = -\frac{1}{2}(A + B) + \frac{i\sqrt{3}}{2}(A - B), t_3 = -\frac{1}{2}(A + B) - \frac{i\sqrt{3}}{2}(A - B)$, being $A = (\frac{1}{2} + \sqrt{\Delta})^{\frac{1}{3}}, B = (\frac{1}{2} - \sqrt{\Delta})^{\frac{1}{3}}$ and $\Delta = \frac{1}{4} - \frac{x^6}{27}$.

From this analysis we can obtain Binet's formula with the Theorem 2.2.

Theorem 2.2. *The Binet formula of the Perrin polynomial sequence is expressed by:*

$$p_n(x) = C_1 t_1^n + C_2 t_2^n + C_3 t_3^n,$$

where C_1, C_2, C_3 are the coefficients and t_1, t_2, t_3 the roots of the equation $t^3 - x^2t - 1 = 0$.

Proof. According to Definition 2 and the characteristic equation, we can solve the linear system to obtain the values of the coefficients C_1, C_2, C_3 .

Furthermore, we have that the discriminant $\Delta = \frac{1}{4} - \frac{x^6}{27}$, refers to the characteristic equation, being able to identify the shape of the roots. So when $\Delta \neq 0$, all roots will be distinct, thus obtaining $x^6 \neq \frac{27}{4}$. Therefore, if this condition does not exist, the reported coefficients cannot exist, since the roots will be the same, without the Binet formula. \square

Theorem 2.3. *The generating function of the Perrin polynomial sequence, for $n \in \mathbb{N}$, is given by:*

$$g(t) = \sum_{n=0}^{\infty} p_n(x)t^n = \frac{3 - t^2}{(1 - x^2t^2 - t^3)}.$$

Proof. Let $g(t)$ be the generating function of the Perrin polynomial sequence $p_n(x)$, then:

$$\begin{aligned} g(t) - g(t)x^2t^2 - g(t)t^3 &= p_0(x) + p_1(x)t + (p_2(x) - p_0(x))t^2, \\ g(t)(1 - x^2t^2 - t^3) &= 3 - t^2, \\ g(t) &= \frac{3 - t^2}{(1 - x^2t^2 - t^3)}. \end{aligned} \quad \square$$

3 Perrin's complex bivariate polynomials

In this section, Perrin's complex bivariate polynomials will be introduced, based on the work of [3].

Definition 3. *Perrin's complex bivariate polynomials $p_n(x, y)$, for $n \geq 3$, are given by the formula:*

$$p_n(x, y) = ix^2p_{n-2}(x, y) + y^2p_{n-3}(x, y),$$

on what $p_0(x, y) = 3, p_1(x, y) = 0, p_2(x, y) = 2$ and $i^2 = -1$.

Thus, we have the first terms of this sequence presented in the Table 2 below.

n	$p_n(x, y)$
0	3
1	0
2	2
3	$3y^2$
4	$2x^2i$
5	$2y^2 + 3x^2y^2i$
6	$-2x^4 + 3y^4$
\vdots	\vdots

Table 2. First terms of Perrin's complex bivariate polynomial sequence

Furthermore, some properties, inherent to this complex polynomial sequence, are discussed, investigating its matrix form, generating function and Binet formula.

Theorem 3.1. *The generating matrix of Perrin's complex bivariate polynomials, for $n \geq 1$ and with $n \in \mathbb{N}$, is given by:*

$$uQ_c^n = \begin{bmatrix} 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ ix^2 & 0 & 1 \\ y^2 & 0 & 0 \end{bmatrix}^n = \begin{bmatrix} p_{n+2}(x) & p_{n+1}(x) & p_n(x) \end{bmatrix}$$

where u represents the vector with the sequence initialization values, and Q_c represents the complex generating matrix.

Proof. Through the principle of finite induction, we have the following. For $n = 1$, we have that:

$$vQ_c^1 = \begin{bmatrix} 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ ix^2 & 0 & 1 \\ y^2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3y^2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} p_3(x) & p_2(x) & p_1(x) \end{bmatrix}$$

We validate the equality discussed above. Assuming it is valid for $n = k, k \in \mathbb{N}$, we have that:

$$vQ_c^k = \begin{bmatrix} 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ ix^2 & 0 & 1 \\ y^2 & 0 & 0 \end{bmatrix}^k = \begin{bmatrix} p_{k+2}(x) & p_{k+1}(x) & p_k(x) \end{bmatrix}$$

Now, verifying that it is valid for $n = k + 1$, we have that:

$$\begin{aligned} vQ_c^{k+1} &= \begin{bmatrix} 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ ix^2 & 0 & 1 \\ y^2 & 0 & 0 \end{bmatrix}^k \begin{bmatrix} 0 & 1 & 0 \\ ix^2 & 0 & 1 \\ y^2 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} p_{k+2}(x) & p_{k+1}(x) & p_k(x) \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ ix^2 & 0 & 1 \\ y^2 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} ix^2p_{k+1}(x) + y^2p_k(x) & p_{k+2}(x) & p_{k+1}(x) \end{bmatrix} \\ &= \begin{bmatrix} p_{k+3}(x) & p_{k+2}(x) & p_{k+1}(x) \end{bmatrix}. \quad \square \end{aligned}$$

The characteristic equation of this complex bivariate Perrin polynomial sequence is given by $q^3 - ix^2q - y^2 = 0$, on what x^2 and y^2 are the polynomial variables and i the imaginary component. So, we have that q_1, q_2 and q_3 are the roots of the characteristic equation, with: $q_1 = C + D$, $q_2 = -\frac{1}{2}(C + D) + \frac{i\sqrt{3}}{2}(C - D)$, $q_3 = -\frac{1}{2}(C + D) - \frac{i\sqrt{3}}{2}(C - D)$, having $C = (\frac{y^2}{2} + \sqrt{\Delta})^{\frac{1}{3}}$, $D = (\frac{y^2}{2} - \sqrt{\Delta})^{\frac{1}{3}}$ and $\Delta = \frac{y^4}{4} - \frac{i^3x^6}{27}$.

From this analysis we can obtain Binet's formula with the Theorem 3.2.

Theorem 3.2. *The Binet formula of the Perrin polynomial sequence is expressed by:*

$$p_n(x, y) = K_1q_1^n + K_2q_2^n + K_3q_3^n$$

on what K_1, K_2, K_3 are the coefficients and q_1, q_2, q_3 the roots of the equation $q^3 - ix^2q - y = 0$.

Proof. Similar to Theorem 2.2, we have the Definition 2 and the characteristic equation, we can solve the linear system to obtain the values of the coefficients K_1, K_2 and K_3 .

Furthermore, we have that the discriminant $\Delta = \frac{y^2}{4} - \frac{i^3x^6}{27}$, refers to the characteristic equation, being able to identify the shape of the roots. So when $\Delta \neq 0$, all roots will be distinct, thus obtaining $ix^6 \neq \frac{27y^2}{4}$. Thus, if this condition does not exist, the coefficients cannot exist. Thus, there is no Binet formula. \square

Theorem 3.3. *The generating function of Perrin's complex bivariate polynomial sequence, for $n \in \mathbb{N}$, is given by:*

$$g_c(t) = \sum_{n=0}^{\infty} p_n(x, y)t^n = \frac{3 - t^2}{(1 - ix^2t^2 - y^2t^3)}.$$

Proof. Let $g_c(t)$ be the generating function of the complex bivariate polynomial sequence of Perrin $p_n(x, y)$, then:

$$\begin{aligned} g_c(t) - g_c(t)ix^2t^2 - g_c(t)y^2t^3 &= p_0(x, y) + p_1(x, y)t + (p_2(x, y) - p_0(x, y))t^2, \\ g_c(t)(1 - ix^2t^2 - y^2t^3) &= 3 - t^2, \\ g_c(t) &= \frac{3 - t^2}{(1 - ix^2t^2 - y^2t^3)}. \end{aligned} \quad \square$$

4 Perrin complex bivariate polynomials for non-positive integer indices

We emphasize that the main result of this work, is presented in this section, reporting the generalization of Perrin's complex bivariate numbers, performing an extension to the field of numbers with non-positive integer indices.

Extending to the field of non-positive integers, we then have Perrin's complex bivariate polynomials for non-positive integer indices. With that, we have the first terms presented in Table 3 below.

$-n$	$p_{-n}(x, y)$
\vdots	\vdots
-6	$\frac{-6x^4+12x^6i}{y^8} + \frac{-2x^{10}i-3x^{12}+3y^8}{y^{12}}$
-5	$\frac{-4x^2i-9x^4}{y^6} + \frac{2x^8-3x^{10}i}{y^{10}}$
-4	$\frac{2-6x^2i}{y^4} + \frac{2x^6+3x^8}{y^8}$
-3	$\frac{-2x^4i+3x^6i}{y^6}$
-2	$\frac{-2x^2i-3x^4}{y^4}$
-1	$\frac{2-3x^2i}{y^2}$
0	3

Table 3. First negative terms of Perrin's complex bivariate polynomial sequence

Theorem 4.1. *The matrix form of Perrin's complex bivariate sequence for non-positive integer indices, with $n > 0$ and $n \in \mathbb{N}$, is given by:*

$$uQ_c^{-n} = \begin{bmatrix} 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{1}{y^2} \\ 1 & 0 & 0 \\ 0 & 1 & -\frac{ix^2}{y^2} \end{bmatrix}^{-n} = \begin{bmatrix} p_{-n+2}(x) & p_{-n+1}(x) & p_{-n}(x) \end{bmatrix}$$

where u represents the vector with the sequence initialization values, and Q_c^{-1} represents the complex generating matrix inverse.

Proof. Through the principle of finite induction, we have the following. For $n = 1$, we have that:

$$vQ_c^{-1} = \begin{bmatrix} 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{1}{y^2} \\ 1 & 0 & 0 \\ 0 & 1 & -\frac{ix^2}{y^2} \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 3 & \frac{2}{y^2} - \frac{3x^2i}{y^4} \end{bmatrix} = \begin{bmatrix} p_1(x) & p_0(x) & p_{-1}(x) \end{bmatrix}.$$

We validate the equality discussed above. Assuming it is valid for $n = k, k \in \mathbb{N}$, we have that:

$$vQ_c^{-k} = \begin{bmatrix} 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{1}{y^2} \\ 1 & 0 & 0 \\ 0 & 1 & -\frac{ix^2}{y^2} \end{bmatrix}^{-k} = \begin{bmatrix} p_{-k+2}(x) & p_{-k+1}(x) & p_{-k}(x) \end{bmatrix}.$$

Now, verifying that it is valid for $n = k + 1$, we have that:

$$\begin{aligned} vQ_c^{-(k+1)} &= \begin{bmatrix} 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ ix^2 & 0 & 1 \\ y^2 & 0 & 0 \end{bmatrix}^{-k} \begin{bmatrix} 0 & 1 & 0 \\ ix^2 & 0 & 1 \\ y^2 & 0 & 0 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} p_{-k+2}(x) & p_{-k+1}(x) & p_{-k}(x) \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{1}{y^2} \\ 1 & 0 & 0 \\ 0 & 1 & -\frac{ix^2}{y^2} \end{bmatrix}^{-1} \\ &= \begin{bmatrix} p_{-k+1}(x) & p_{-k}(x) & \frac{p_{-k+2}(x) - x^2ip_{-k}(x)}{y^2} \end{bmatrix} \\ &= \begin{bmatrix} p_{-k+1}(x) & p_{-k}(x) & p_{-k-1}(x) \end{bmatrix}. \quad \square \end{aligned}$$

Theorem 4.2. *The generating function of Perrin's complex bivariate sequence for non-positive integer indices is given by:*

$$g_{cn}(t) = \sum_{n=0}^{\infty} p_{-n}(x, y)t^n = \frac{(1+t)\left(\frac{-3y^2-2t+3x^2it}{y^2}\right) + (2i+3x^2)\left(\frac{x^2t^2}{y^4}\right)}{(y^2t^3 - t - 1)}.$$

Proof. Let $g_{cn}(t)$ be the generating function of the complex bivariate polynomial sequence of Perrin $p_n(x, y)$, then:

$$g_{cn}(t)y^2t^3 - g_{cn}(t)t - g_{cn} = -p_0(x, y)t - (p_0(x, y) + p_{-1}(x, y))t - (p_{-1}(x, y) + p_{-2}(x, y))t^2,$$

$$g_{cn}(t)(y^2t^3 - t - 1) = (1+t)\left(\frac{-3y^2 - 2t + 3x^2it}{y^2}\right) + (2i + 3x^2)\left(\frac{x^2t^2}{y^4}\right),$$

$$g_{cn}(t) = \frac{(1+t)\left(\frac{-3y^2-2t+3x^2it}{y^2}\right) + (2i+3x^2)\left(\frac{x^2t^2}{y^4}\right)}{(y^2t^3 - t - 1)}. \quad \square$$

5 Conclusion

This work presents a study around Perrin's sequence presenting its polynomial form and its complex bivariate polynomial form, in which, its numbers were worked on functions of variables and explored in its complex form after the insertion of the imaginary component i . It was possible to present the recurrence of these numbers, their generating matrix, Binet formula and generating function.

We reinforce the presentation, in this paper, of the investigation of Perrin's complex bivariate numbers, with positive and non-positive integer indices, based on the work of Catalani [3], which deals with the Fibonacci bivariate form.

For future work, we propose investigations around these complex bivariate polynomials and polynomial numbers, applying this mathematical content to other areas.

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