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Relations between R_α, R_β and R_m functions related to Jacobi's triple-product identity and the family of theta-function identities

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Abstract: In this paper, the author establishes a set of three new theta-function identities involving R_α, R_β and R_m functions which are based upon a number of q-product identities and Jacobi's celebrated triple-product identity. These theta-function identities depict the inter-relationships that exist among theta-function identities and combinatorial partition-theoretic identities. Here, in this paper we answer to an open question of Srivastava *et al* [33], and estabish relations in terms of R_α, R_β and R_m (for $m = 1, 2, 3$); and q-products identities. Finally, we choose to further emphasize upon some close connections with combinatorial partition-theoretic identities.
Keywords: Theta-function identities, Multivariable R-functions, Jacobi's triple-product identity, Ramanujan's theta functions, q-Product identities, Euler's Pentagonal Number Theorem, Rogers–Ramanujan continued fraction, Rogers–Ramanujan identities, Combinatorial partition-theoretic identities.
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1 Introduction

Recently, Srivastava *et al* [33] coined an open problem, which state as find an inter-relationships between R_α, R_α and $R_m(m \in N)$, q-product identities and continued-fraction identities. The purpose of this article is to establish relationships between R_α, R_α and $R_m(m \in N)$ and the q-product identities. Throughout this article, we denote by $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{C}$ the set of positive integers, the set of integers and the set of complex numbers, respectively. We also let

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$$

and recall the following q-notations (see, for example, [35, Chapter 6] and [36, pp. 346 *et seq.*]). The q-shifted factorial $(a; q)_n$ is defined (for $|q| < 1$) by

$$(a; q)_n := \begin{cases} 1 & (n = 0) \\ \prod_{k=0}^{n-1}(1 - aq^k) & (n \in \mathbb{N}), \end{cases} \tag{1}$$

where $a, q \in \mathbb{C}$ and it is assumed *tacitly* that $a \neq q^{-m} \quad (m \in \mathbb{N}_0)$. We also write

$$(a; q)_\infty := \prod_{k=0}^{\infty}(1 - aq^k) = \prod_{k=1}^{\infty}(1 - aq^{k-1}) \qquad (a, q \in \mathbb{C};\ |q| < 1). \tag{2}$$

It should be noted that, when $a \neq 0$ and $|q| \geqq 1$, the infinite product in the equation (2) diverges. So, whenever $(a; q)_\infty$ is involved in a given formula, the constraint $|q| < 1$ will be *tacitly* assumed to be satisfied.

The following notations are also frequently used in our investigation:

$$(a_1, a_2, \ldots, a_m; q)_n := (a_1; q)_n\ (a_2; q)_n\ \cdots\ (a_m; q)_n \tag{3}$$

and

$$(a_1, a_2, \ldots, a_m; q)_\infty := (a_1; q)_\infty\ (a_2; q)_\infty\ \cdots\ (a_m; q)_\infty. \tag{4}$$

Ramanujan (see [27,28]) defined the general theta function $\mathfrak{f}(a, b)$ as follows (see, for details, [5, p. 31, Eq. (18.1)] and [30]):

$$\begin{aligned} \mathfrak{f}(a, b) &= 1 + \sum_{n=1}^{\infty}(ab)^{\frac{n(n-1)}{2}}\ (a^n + b^n) \\ &= \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}}\ b^{\frac{n(n-1)}{2}} = \mathfrak{f}(b, a) \qquad (|ab| < 1). \end{aligned} \tag{5}$$

We find from this last equation (5) that

$$\mathfrak{f}(a, b) = a^{\frac{n(n+1)}{2}}\ b^{\frac{n(n-1)}{2}}\ \mathfrak{f}\big(a(ab)^n, b(ab)^{-n}\big) = \mathfrak{f}(b, a) \qquad (n \in \mathbb{Z}). \tag{6}$$

In fact, Ramanujan (see [27,28]) also rediscovered Jacobi's famous triple-product identity which, in Ramanujan's notation, is given by (see [5, p. 35, Entry 19]):

$$\mathfrak{f}(a, b) = (-a; ab)_\infty\ (-b; ab)_\infty\ (ab; ab)_\infty \tag{7}$$

or, equivalently, by (see [25])

$$\begin{aligned} \sum_{n=-\infty}^{\infty} q^{n^2}\ z^n &= \prod_{n=1}^{\infty}\left(1 - q^{2n}\right)\left(1 + zq^{2n-1}\right)\left(1 + \frac{1}{z}\ q^{2n-1}\right) \\ &= \left(q^2; q^2\right)_\infty\ \left(-zq; q^2\right)_\infty\ \left(-\frac{q}{z}; q^2\right)_\infty \qquad (|q| < 1;\ z \neq 0). \end{aligned}$$

Several q-series identities, which emerge naturally from Jacobi's triple-product identity (7), are worth noting here (see, for details, [5, pp. 36–37, Entry 22]):

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2\sum_{n=1}^{\infty} q^{n^2}$$
$$= \left\{(-q;q^2)_\infty\right\}^2 (q^2;q^2)_\infty = \frac{(-q;q^2)_\infty \; (q^2;q^2)_\infty}{(q;q^2)_\infty \; (-q^2;q^2)_\infty}; \tag{8}$$
$$\psi(q) := \mathfrak{f}(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}; \tag{9}$$
$$f(-q) := \mathfrak{f}(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n \, q^{\frac{n(3n-1)}{2}}$$
$$= \sum_{n=0}^{\infty}(-1)^n \, q^{\frac{n(3n-1)}{2}} + \sum_{n=1}^{\infty}(-1)^n \, q^{\frac{n(3n+1)}{2}} = (q;q)_\infty. \tag{10}$$

Equation (10) is known as Euler's *Pentagonal Number Theorem*. Remarkably, the following q-series identity:

$$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty} = \frac{1}{\chi(-q)} \tag{11}$$

provides the analytic equivalent form of Euler's famous theorem (see, for details, [2] and [4]).

Note: Equation (6) holds true as stated only if n is any integer. Moreover, in case n is not an integer, this result (6) is only approximately true (see, for details, [27, Vol. 2, Chapter XVI, p. 193, Entry 18 (iv)]). Historically speaking, the q-series identity (7) or its above-mentioned equivalent form was first proved by Carl Friedrich Gauss (1777–1855).

Theorem 1.1 (Euler's Pentagonal Number Theorem)**.** *The number of partitions of a given positive integer n into distinct parts is equal to the number of partitions of n into odd parts.*

We also recall the Rogers–Ramanujan continued fraction $R(q)$ given by

$$R(q) := q^{\frac{1}{5}} \frac{H(q)}{G(q)} = q^{\frac{1}{5}} \frac{\mathfrak{f}(-q,-q^4)}{\mathfrak{f}(-q^2,-q^3)} = q^{\frac{1}{5}} \frac{(q;q^5)_\infty \; (q^4;q^5)_\infty}{(q^2;q^5)_\infty \; (q^3;q^5)_\infty}$$
$$= \frac{q^{\frac{1}{5}}}{1+} \; \frac{q}{1+} \; \frac{q^2}{1+} \; \frac{q^3}{1+} \qquad (|q| < 1). \tag{12}$$

Here $G(q)$ and $H(q)$, which are associated with the widely-investigated Roger–Ramanujan identities, are defined as follows:

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{f(-q^5)}{\mathfrak{f}(-q,-q^4)}$$
$$= \frac{1}{(q;q^5)_\infty \; (q4;q^5)_\infty} = \frac{(q^2;q^5)_\infty \; (q^3;q^5)_\infty \; (q^5;q^5)_\infty}{(q;q)_\infty} \tag{13}$$

and

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{f(-q^5)}{\mathfrak{f}(-q^2,-q^3)} = \frac{1}{(q^2;q^5)_\infty \; (q^3;q^5)_\infty}$$
$$= \frac{(q;q^5)_\infty \; (q^4;q^5)_\infty \; (q^5;q^5)_\infty}{(q;q)_\infty}, \tag{14}$$

and the functions $\mathfrak{f}(a,b)$ and $f(-q)$ are given by the equations (5) and (10), respectively.

For a detailed historical account of (and for various related developments stemming from) the Rogers–Ramanujan continued fraction (12), as well as the Rogers–Ramanujan identities (13) and (14), the interested reader may refer to the monumental work [5, p. 77 *et seq.*] (see also [30] and [35]).

The following continued-fraction results may be recalled now (see, for example, [8, p. 5, Eq. (2.8)]).

Theorem 1.2. *Suppose that* $|q| < 1$. *Then*

$$(q^2;q^2)_\infty(-q;q)_\infty = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{1}{1-}\,\frac{q}{1+}\,\frac{q(1-q)}{1-}\,\frac{q^3}{1+}\,\frac{q^2(1-q^2)}{1-}\,\frac{q^5}{1+}\,\frac{q^3(1-q^3)}{1-}\cdots$$

$$= \cfrac{1}{1-\cfrac{q}{1+\cfrac{q(1-q)}{1-\cfrac{q^3}{1+\cfrac{q^2(1-q^2)}{1-\cfrac{q^5}{1+\cfrac{q^3(1-q^3)}{1-\vdots}}}}}}}, \tag{15}$$

$$\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty} = \frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^4}{1+}\,\frac{q^5}{1+}\,\frac{q^6}{1+}\cdots$$

$$= \cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cfrac{q^4}{1+\cfrac{q^5}{1+\cfrac{q^6}{1+\vdots}}}}}}}, \tag{16}$$

and

$$C(q) = \frac{(q^2;q^5)_\infty(q^3;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty} = 1 + \frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^4}{1+}\,\frac{q^5}{1+}\,\frac{q^6}{1+}\cdots$$

$$= 1 + \cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cfrac{q^4}{1+\cfrac{q^5}{1+\cfrac{q^6}{1+\vdots}}}}}} \tag{17}$$

By introducing the general family $R(s,t,l,u,v,w)$, Andrews *et al.* [3] investigated a number of interesting double-summation hypergeometric q-series representations for several families of partitions and further explored the role of double series in combinatorial-partition identities:

$$R(s,t,l,u,v,w) := \sum_{n=0}^{\infty} q^{s\binom{n}{2}+tn}\, r(l,u,v,w;n), \tag{18}$$

where

$$r(l,u,v,w:n) := \sum_{j=0}^{[\frac{n}{u}]} (-1)^j \,\frac{q^{uv\binom{j}{2}+(w-ul)j}}{(q;q)_{n-uj}\,(q^{uv};q^{uv})_j}. \tag{19}$$

We also recall the following interesting special cases of (18) (see, for details, [3, p. 106, Theorem 3]; see also [30]):

$$R(2,1,1,1,2,2) = (-q;q^2)_\infty, \tag{20}$$

$$R(2,2,1,1,2,2) = (-q^2;q^2)_\infty \tag{21}$$

and

$$R(m,m,1,1,1,2) = \frac{(q^{2m};q^{2m})_\infty}{(q^m;q^{2m})_\infty}. \tag{22}$$

Recently, Srivastava *et al.* (see [33]) has introduced three notations:

$$R_\alpha = R(2,1,1,1,2,2);\, R_\beta = R(2,2,1,1,2,2);\, R_m = R(m,m,1,1,1,2);\, m = 1,2,3,\ldots. \tag{23}$$

for multivarite R-functions, which we shall use for computation of our main results in Section 2.

Ever since the year 2015, several new advancements and generalizations of the existing results were made in regard to combinatorial partition-theoretic identities (see, for example, [9–23] and [30–32]). In particular, Chaudhary *et al.* generalized several known results on character formulas (see [20]), Roger-Ramanujan type identities (see [17]), Eisenstein series, the Ramanujan–Göllnitz–Gordon continued fraction (see [18]), the 3-dissection property (see [14]), Ramanujan's modular equations of degrees 3, 7 and 9 (see [11, 13]), and so on, by using combinatorial partition-theoretic identities. An interesting recent investigation on the subject of combinatorial partition-theoretic identities by Hahn *et al.* [24] is also worth mentioning in this connection.

Here, in this paper, our main objective is to establish a set of three new theta-function identities which depict the inter-relationships in terms of R_α, R_β and R_m functions along with q-product identities.

Each of the following preliminary results will be needed for the demonstration of our main results in this paper (see [1, Theorem 5.1; 7, Entry 51, p. 204; 27, Theorem 3.1]):

[A]. If

$$U = \frac{\phi^4(-q)}{\phi^4(-q^3)} \quad \text{and} \quad V = \frac{\psi^4(q)}{\psi^4(q^3)},$$

then

$$U - UV + V - 9 = 0. \tag{24}$$

[B]. If

$$M = \frac{f^2(-q)}{q^{\frac{1}{6}} f^2(-q^3)} \quad \text{and} \quad N = \frac{f^2(-q^2)}{q^{\frac{1}{3}} f^2(-q^6)},$$

then

$$MN + \frac{9}{MN} - \left(\frac{N}{M}\right)^3 - \left(\frac{M}{N}\right)^3 = 0. \tag{25}$$

[C]. If

$$U=\frac{\phi^4(q)}{\phi^4(q^3)},$$

then

$$\frac{f^4(q)f^4(-q^2)}{qf^4(q^3)f^4(-q^6)}-\frac{U(U-9)}{(1-U)}=0,\ \ U\neq 1. \tag{26}$$

2 A set of main results

In this section, we state and prove a set of three identities which depict inter-relationships among q-product identities; R_α, R_β, and R_m.

2.1 Main results

Theorem 3. *Each of the following relationships holds true:*

$$\frac{1}{q}\left\{\frac{R_1}{R_3}\right\}^4+\left(1-\frac{1}{q}\left\{\frac{R_1}{R_3}\right\}^4\right)\cdot \\ \cdot\frac{(q,q,q,q,q^2,q^2,q^2,q^2;q^2)_\infty(-q^3,-q^3,-q^3,-q^3,-q^6,-q^6,-q^6,-q^6;q^6)_\infty}{\{R_\alpha R_\beta\}^4(q^3,q^3,q^3,q^3,q^6,q^6,q^6,q^6;q^6)_\infty}=9, \tag{27}$$

Equation (27) give inter-relationships between R_1, R_3, R_α and R_β.

$$\frac{(q,q,q^2,q^2,q^2,q^2;q^2)_\infty}{q^{\frac{1}{2}}(q^3,q^3,q^6,q^6,q^6,q^6;q^6)_\infty}+\frac{9q^{\frac{1}{2}}(q^3,q^3,q^6,q^6,q^6,q^6;q^6)_\infty}{(q,q,q^2,q^2,q^2,q^2;q^2)_\infty}- \\ -\frac{1}{q^{\frac{1}{2}}}\left\{\frac{R_1R_{12}(q^3;q^6)_\infty(q^6;q^{12})_\infty(q^{12};q^{24})_\infty}{(q^2;q^2)_\infty(q^6;q^{12})_\infty(q^{24};q^{24})_\infty}\right\}^6-q^{\frac{1}{2}}\left\{\frac{R_2R_3(q;q^2)_\infty(q^2;q^4)_\infty}{(q^4;q^4)_\infty(q^6;q^6)_\infty}\right\}^6=0, \tag{28}$$

Equation (28) give inter-relationships between R_1, R_2, R_3 and R_{12}.

$$\frac{(-q,-q,-q,-q;-q)_\infty(q^2,q^2,q^2,q^2;q^2)_\infty}{q\cdot(-q^3,-q^3,-q^3,-q^3;-q^3)_\infty(q^6,q^6,q^6,q^6;q^6)_\infty}-\frac{\{R_\alpha R_1(q^3,-q^6;q^6)_\infty\}^4}{\{(-q^3,q^6;q^6)_\infty\}^4}\cdot \\ \cdot\frac{\{R_1R_\alpha(q^3,-q^6;q^6)_\infty\}^4-9\{R_2(-q^3,q^6;q^6)\}^4}{\{R_\beta(-q^3,q^6;q^6)_\infty\}^4}\cdot \\ \cdot\frac{\{(-q^3,q^6;q^6)_\infty\}^4}{\{\{R_2(-q^3,q^6;q^6)_\infty\}^4-\{R_1R_\alpha(q^3,q^6;q^6)\}^4}=0. \tag{29}$$

Equation (29) gives inter-relationships between R_1, R_2, R_α and R_β.

It is assumed that each member of the assertions (27) to (29) exists.

2.2 Proofs of the main results

First of all, in order to prove the assertion (27), apply the identities (8), (9) and (23) into (24), we obtain;

$$U=\frac{\phi^4(-q)}{\phi^4(-q^3)}=\frac{(q,q,q,q,q^2,q^2,q^2,q^2;q^2)_\infty}{\{R_\alpha R_\beta\}^4}\cdot \\ \cdot\frac{(-q^3,-q^3,-q^3,-q^3,-q^6,-q^6,-q^6,-q^6;-q^6)_\infty}{(q^3,q^3,q^3,q^3,q^6,q^6,q^6,q^6;q^6)_\infty}, \tag{30}$$

$$V=\frac{\psi^4(q)}{q\psi^4(q^3)}=\frac{1}{q}\left\{\frac{R_1}{R_3}\right\}^4, \tag{31}$$

$$UV=\frac{(q,q,q,q,q^2,q^2,q^2,q^2;q^2)_\infty\{R_1\}^4}{\{R_\alpha R_\beta\}^4}\cdot$$
$$\cdot\frac{(-q^3,-q^3,-q^3,-q^3,-q^6,-q^6,-q^6,-q^6;-q^6)_\infty}{q.(q^3,q^3,q^3,q^3,q^6,q^6,q^6,q^6;q^6)_\infty\{R_3\}^4}, \tag{32}$$

Hence, with the help of identities (30) and (32), we obtain

$$U-UV=\left(1-\frac{1}{q}\left\{\frac{R_1}{R_3}\right\}^4\right)\cdot\frac{(q,q,q,q,q^2,q^2,q^2,q^2;q^2)_\infty}{\{R_\alpha R_\beta\}^4}\cdot$$
$$\cdot\frac{(-q^3,-q^3,-q^3,-q^3,-q^6,-q^6,-q^6,-q^6;-q^6)_\infty}{(q^3,q^3,q^3,q^3,q^6,q^6,q^6,q^6;q^6)_\infty}, \tag{33}$$

and with the help of identity (31), we have

$$V-9=\left(\frac{1}{q}\left\{\frac{R_1}{R_3}\right\}^4-9\right). \tag{34}$$

combining the identities (33) and (34), as precondition given in (24), we are led to the first assertion (27).

Next, we prove the second q-series identity (28). Applying the identities (10) and (23) into (25), we obtain:

$$MN+\frac{9}{MN}=\frac{(q,q,q^2,q^2,q^2,q^2;q^2)_\infty}{q^{\frac{1}{2}}(q^3,q^3,q^6,q^6,q^6,q^6;q^6)_\infty}+\frac{9q^{\frac{1}{2}}(q^3,q^3,q^6,q^6,q^6,q^6;q^6)_\infty}{q^{\frac{1}{2}}(q,q,q^2,q^2,q^2,q^2;q^2)_\infty}, \tag{35}$$

and

$$\left(\frac{N}{M}\right)^3+\left(\frac{M}{N}\right)^3=\frac{1}{q^{\frac{1}{2}}}\left\{\frac{R_1R_{12}(q^3;q^6)_\infty(q^6;q^{12})_\infty(q^{12};q^{24})_\infty}{(q^2;q^2)_\infty(q^6;q^{12})_\infty(q^{24};q^{24})_\infty}\right\}^6$$
$$+q^{\frac{1}{2}}\left\{\frac{R_\alpha R_3(q;q^2)_\infty(q^2;q^4)_\infty}{(q^4;q^4)_\infty(q^6;q^6)_\infty}\right\}^6. \tag{36}$$

combining the identities (35) and (36), as precondition given in (25), we complete our demonstration of assertion (28).

Finally, we attempt to prove the third q-series identity (29). Applying the identities (8), (10) and (23) into (25), we get;

$$\frac{U^2-9U}{(1-U)}=\frac{U(U-9)}{(1-U)}=\frac{\{R_\alpha R_1(q^3,-q^6;q^6)_\infty\}^4}{\{(-q^3,q^6;q^6)_\infty\}^4}\cdot$$
$$\cdot\frac{\{R_1R_\alpha(q^3,-q^6;q^6)_\infty\}^4-9\{R_2(-q^3,q^6;q^6)\}^4}{\{R_\beta(-q^3,q^6;q^6)_\infty\}^4}\cdot$$
$$\cdot\frac{\{(-q^3,q^6;q^6)_\infty\}^4}{\{R_2(-q^3,q^6;q^6)_\infty\}^4-\{R_1R_\alpha(q^3,-q^6;q^6)\}^4}, \tag{37}$$

and

$$\frac{f^4(q)f^4(-q^2)}{qf^4(q^3)f^4(-q^6)}=\frac{(-q,-q,-q,-q;-q)_\infty(q^2,q^2,q^2,q^2;q^2)_\infty}{q.(-q^3,-q^3,-q^3,-q^3;-q^3)_\infty(q^6,q^6,q^6,q^6;q^6)_\infty}. \tag{38}$$

combining (37) and (38), as precondition given in (26), we complete our demonstration of assertion (29).

We thus have completed our proof of the above Theorem.

3 Connections with combinatorial partition-theoretic identities

Several extensions and generalizations of partition-theoretic identities and other q-identities, which we have investigated in this paper, as well as their connections with combinatorial partition-theoretic identities, can be found in several recent works (see, for example, [26], [38] and [39]). The demonstrations in some of these recent developments are also based upon their combinatorial interpretations and generating functions (see also [24]).

On the connections of different partition-theoretic identities, several findings and observations had been made by the researchers. But, recently valuable progress in this direction has been made by Andrews *et al.* (see [3]); and established three results for the double series associated with Schur's partitions, Göllnitz–Gordon partitions and Göllnitz partitions in terms of multivariate R-functions. Further, Srivastava et al (see [33]) has generalized multivariate R-functions in terms of R_α, R_β and R_m.

4 Concluding remarks and observations

In this article, we have established three new relationships among R_α, R_β and R_m functions related to Jacobi's triple-product identity and the family of theta-function identities, which were motivated by several recent developments dealing essentially with theta-function identities and combinatorial partition-theoretic identities. Here, in this article, we have established a family of three presumably new theta-function identities which depict the inter-relationships that exist among q-product identities and multivariate R-functions. We have also considered several closely related identities such as (for example) q-product identities and Jacobi's triple-product identities. And, with a view to further motivating researches involving theta-function identities and combinatorial partition-theoretic identities, we have chosen to indicate rather briefly a number of recent developments on the subject-matter of this article. The list of citations, which we have included in this article, is believed to be potentially useful for indicating some of the directions for further researches and related developments on the subject-matter which we have dealt with here. In particular, the recent works by Cao *et al.* [6], Chaudhary *et al.* (see [8], [19–23]), Hahn *et al.* [24], and Srivastava *et al.* (see [29, 31, 34, 37–39]) are worth mentioning here.

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