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On the connections between Pell numbers and Fibonacci p-numbers

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Abstract: In this paper, we define the Fibonacci–Pell p-sequence and then we discuss the connection of the Fibonacci–Pell p-sequence with the Pell and Fibonacci p-sequences. Also, we provide a new Binet formula and a new combinatorial representation of the Fibonacci–Pell p-numbers by the aid of the n-th power of the generating matrix of the Fibonacci–Pell p-sequence. Furthermore, we derive relationships between the Fibonacci–Pell p-numbers and their permanent, determinant and sums of certain matrices.

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1 Introduction

The well-known Pell sequence $\{P_n\}$ is defined by the following recurrence relation:

$$P_{n+2} = 2P_{n+1} + P_n \text{ for } n \geq 0 \text{ in which } P_0 = 0 \text{ and } P_1 = 1.$$

There are many important generalizations of the Fibonacci sequence. The Fibonacci p-sequence [22, 23] is one of them:

$$F_p(n) = F_p(n-1) + F_p(n-p-1) \text{ for } p = 1, 2, 3, \dots \text{ and } n > p$$

in which $F_p(0) = 0$, $F_p(1) = \cdots = F_p(p) = 1$. When $p = 1$, the Fibonacci p-sequence $\{F_p(n)\}$ is reduced to the usual Fibonacci sequence $\{F_n\}$.

It is easy to see that the characteristic polynomials of the Pell sequence and Fibonacci p-sequence are $f_1(x) = x^2 - 2x - 1$ and $f_2(x) = x^{p+1} - x^p - 1$, respectively. We use these in the next section.

Let the $(n+k)$-th term of a sequence be defined recursively by a linear combination of the preceding k terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1},$$

in which $c_0, c_1, \dots, c_{k-1}$ are real constants. In [12], Kalman derived a number of closed-form formulas for the generalized sequence by the companion matrix method as follows:

Let the matrix A be defined by

$$A = [a_{i,j}]_{k \times k} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & \cdots & c_{k-2} & c_{k-1} \end{bmatrix},$$

then

$$A^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}$$

for $n \geq 0$.

Several authors have used homogeneous linear recurrence relations to deduce miscellaneous properties for a plethora of sequences: see for example, [1, 4, 8–11, 19–21, 24]. In [5–7, 14–16, 22, 23, 25], the authors defined some linear recurrence sequences and gave their various properties by matrix methods.

In the present paper, we discuss connections between the Pell and Fibonacci p-numbers. Firstly, we define the Fibonacci–Pell p-sequence and then we study recurrence relation among this sequence, Pell and Fibonacci p-sequences. In addition, we obtain their generating matrices, Binet formulas, permanental, determinantal, combinatorial, exponential representations, and we derive a formula for the sums of the Fibonacci–Pell p-numbers.

2 Main results

Now we define the Fibonacci–Pell p-sequence $\left\{F_n^{P,p}\right\}$ by the following homogeneous linear recurrence relation for any given $p\,(3,4,5,\ldots)$ and $n \geq 0$

$$F_{n+p+3}^{P,p} = 3F_{n+p+2}^{P,p} - F_{n+p+1}^{P,p} - F_{n+p}^{P,p} + F_{n+2}^{P,p} - 2F_{n+1}^{P,p} - F_n^{P,p}, \tag{1}$$

in which $F_0^{P,p} = \cdots = F_{p+1}^{P,p} = 0$ and $F_{p+2}^{P,p} = 1$.

First, we consider the relationship between the Fibonacci–Pell p-sequence which is defined above, Pell, and Fibonacci p-sequences.

Theorem 2.1. *Let P_n, $F_3(n)$ and $F_n^{P,3}$ be the n-th Pell number, Fibonacci 3-number, and Fibonacci–Pell 3-numbers, respectively. Then, for $n \geq 0$*

$$P_{n+2} = F_{n+5}^{P,3} + 2F_{n+3}^{P,3} + F_3(n+2) + F_3(n).$$

Proof. The assertion may be proved by induction on n. It is clear that

$$P_2 = F_5^{P,3} + 2F_3^{P,3} + F_3(2) + F_3(0) = 2.$$

Suppose that the equation holds for $n \geq 1$. Then we must show that the equation holds for $n+1$. Since the characteristic polynomial of Fibonacci–Pell p-sequence $\left\{F_n^{P,p}\right\}$, is

$$g(x) = x^{p+3} - 3x^{p+2} + x^{p+1} + x^p - x^2 + 2x + 1$$

and

$$g(x) = f_1(x)\, f_2(x),$$

where $f_1(x)$ and $f_2(x)$ are the characteristic polynomials of Pell sequence and Fibonacci p-sequence, respectively, we obtain the following relations:

$$P_{n+6} = 3P_{n+5} - P_{n+4} - P_{n+3} + P_{n+2} - 2P_{n+1} - P_n$$

and

$$F_3(n+6) = 3F_3(n+5) - F_3(n+4) - F_3(n+3) + F_3(n+2) - 2F_3(n+1) - F_3(n)$$

for $n \geq 1$. Thus, the conclusion is obtained. □

Theorem 2.2. *Let P_n and $F_n^{P,p}$ be the n-th Pell number and Fibonacci–Pell p-numbers. Then, for $n \geq 0$ and $p \geq 3$.*

i. Let p be a positive integer, then

$$P_n = F_{n+p+1}^{P,p} - F_{n+p}^{P,p} - F_n^{P,p}.$$

ii. If p is odd, then

$$P_n + P_{n+1} = F_{n+p+2}^{P,p} - F_{n+p}^{P,p} - F_{n+1}^{P,p} - F_n^{P,p}$$

and

iii. If p is odd, then

$$\sum_{i=0}^{n} \left(F_i^{P,p} + P_i\right) = F_{n+p+1}^{P,p}.$$

Proof. Consider the Case ii. The assertion may be proved by induction on n. Then for $p = 3$, it is clear that $P_0 + P_1 = F_5^{P,3} - F_3^{P,3} - F_1^{P,3} - F_0^{P,3} = 1$. Suppose that the equation holds for $n > 0$. Then we must show that the equation holds for $n + 1$. Since the characteristic polynomial of the Pell sequence $\{P_n\}$, is

$$f_1(x) = x^2 - 2x - 1,$$

we obtain the following relations:

$$P_{n+6} = 3P_{n+5} - P_{n+4} - P_{n+3} + P_{n+2} - 2P_{n+1} - P_n$$

for $n \geq 1$. Now we consider the proof for the case $p > 3$. Suppose that the equation holds for $p = 2\alpha + 1$, $(\alpha \in \mathbb{N})$ and $n \geq 0$, it is clear that

$$P_n + P_{n+1} = F_{n+2\alpha+3}^{P,2\alpha+1} - F_{n+2\alpha+1}^{P,2\alpha+1} - F_{n+1}^{P,2\alpha+1} - F_n^{P,2\alpha+1}.$$

Then we must show that the equation holds for $p = 2\alpha + 3$, $(\alpha \in \mathbb{N})$. For $n = 0$, it is clear that

$$P_0 + P_1 = F_{2\alpha+5}^{P,2\alpha+1} - F_{2\alpha+3}^{P,2\alpha+1} - F_1^{P,2\alpha+1} - F_0^{P,2\alpha+1} = 1.$$

The assertion may be proved again by induction on n. Assume that the equation holds for $n > 0$. Then we must show that the equation holds for $n + 1$. Since the characteristic polynomial of the Pell sequence $\{P_n\}$, is

$$f_1(x) = x^2 - 2x - 1,$$

we obtain the following relations:

$$P_{n+2\alpha+6} = 3P_{n+2\alpha+5} - P_{n+2\alpha+4} - P_{n+2\alpha+3} + P_{n+2} - 2P_{n+1} - P_n$$

for $n \geq 1$. Thus, the conclusion is obtained.

There is a similar proof for Case i and Case iii. $\square$

By the recurrence relation (1), we have

$$\begin{bmatrix} F_{n+p+3}^{P,p} \\ F_{n+p+2}^{P,p} \\ F_{n+p+1}^{P,p} \\ \vdots \\ F_{n+1}^{P,p} \end{bmatrix} = \begin{bmatrix} 3 & -1 & -1 & 0 & \cdots & 0 & 0 & 1 & -2 & -1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+p+2}^{P,p} \\ F_{n+p+1}^{P,p} \\ F_{n+p}^{P,p} \\ \vdots \\ F_n^{P,p} \end{bmatrix}$$

for the Fibonacci–Pell p-sequence $\left\{F_n^{P,p}\right\}$. Letting

$$D_p = \begin{bmatrix} 3 & -1 & -1 & 0 & \cdots & 0 & 0 & 1 & -2 & -1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 \end{bmatrix}_{(p+3)\times(p+3)},$$

the companion matrix $D_p = [d_{i,j}]_{(p+3)\times(p+3)}$ is said to be the Fibonacci–Pell p-matrix. For more details on the companion type matrices, see [17,18]. It can be readily established by mathematical induction that for $p \geq 3$ and $n \geq 3p-1$,

$$(D_p)^n = \begin{bmatrix} F_{n+p+2}^{P,p} & F_p(n-p+1) - F_{n+p+1}^{P,p} - F_{n+p}^{P,p} & F_p(n-p+2) - F_{n+p+1}^{P,p} & F_p(n-p+3) & \cdots & \\ F_{n+p+1}^{P,p} & F_p(n-p) - F_{n+p}^{P,p} - F_{n+p-1}^{P,p} & F_p(n-p+1) - F_{n+p}^{P,p} & F_p(n-p+2) & \cdots & \\ F_{n+p}^{P,p} & F_p(n-p-1) - F_{n+p-1}^{P,p} - F_{n+p-2}^{P,p} & F_p(n-p) - F_{n+p-1}^{P,p} & F_p(n-p+1) & \cdots & D_p^* \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ F_{n+1}^{P,p} & F_p(n-2p) - F_n^{P,p} - F_{n-1}^{P,p} & F_p(n-2p+1) - F_n^{P,p} & F_p(n-2p+2) & \cdots & \\ F_n^{P,p} & F_p(n-2p-1) - F_{n-1}^{P,p} - F_{n-2}^{P,p} & F_p(n-2p) - F_{n-1}^{P,p} & F_p(n-2p+1) & \cdots & \end{bmatrix},$$

where

$$D_p^* = \begin{bmatrix} F_p(n) & F_p(n-p+3) + F_p(n-p) + F_p(n-p-1) + \cdots + F_p(n-2p+3) - F_{n+p+2}^{P,p} & -F_{n+p+1}^{P,p} \\ F_p(n-1) & F_p(n-p+2) + F_p(n-p-1) + F_p(n-p-2) + \cdots + F_p(n-2p+2) - F_{n+p+1}^{P,p} & -F_{n+p}^{P,p} \\ F_p(n-2) & F_p(n-p+1) + F_p(n-p-2) + F_p(n-p-3) + \cdots + F_p(n-2p+1) - F_{n+p}^{P,p} & -F_{n+p-1}^{P,p} \\ \vdots & \vdots & \vdots \\ F_p(n-p-1) & F_p(n-2p+2) + F_p(n-2p-1) + F_p(n-2p-2) + \cdots + F_p(n-3p+2) - F_{n+1}^{P,p} & -F_n^{P,p} \\ F_p(n-p-2) & F_p(n-2p+1) + F_p(n-2p-2) + F_p(n-2p-3) + \cdots + F_p(n-3p+1) - F_n^{P,p} & -F_{n-1}^{P,p} \end{bmatrix}.$$

In [22], Stakhov defined the generalized Fibonacci p-matrix Q_p and derived the n-th power of the matrix Q_p. In [13], Kılıc gave a Binet formula for the Fibonacci p-numbers by matrix method. Now we concentrate on finding another Binet formula for the Fibonacci–Pell p-numbers by the aid of the matrix $(D_p)^n$.

Lemma 2.3. *The characteristic equation of all the Fibonacci–Pell p-numbers*

$$x^{p+3} - 3x^{p+2} + x^{p+1} + x^p - x^2 + 2x + 1 = 0$$

does not have multiple roots for $p \geq 3$.

Proof. It is clear that $x^{p+3}-3x^{p+2}+x^{p+1}+x^p-x^2+2x+1=(x^{p+1}-x^p-1)(x^2-2x-1)$. In [13], it was shown that the equation $x^{p+1}-x^p-1=0$ does not have multiple roots for $p>1$. It is easy to see that the roots of the equation $x^2-2x-1=0$ are $1+\sqrt{2}$ and $1-\sqrt{2}$. Since $\left(1+\sqrt{2}\right)^{p+1}-\left(1+\sqrt{2}\right)^p-1\neq 0$ and $\left(1-\sqrt{2}\right)^{p+1}-\left(1-\sqrt{2}\right)^p-1\neq 0$, the equation $x^{p+3}-3x^{p+2}+x^{p+1}+x^p-x^2+2x+1=0$ does not have multiple roots for $p\geq 3$. □

Let $\alpha_1,\alpha_2,\ldots,\alpha_{p+3}$ be the roots of the equation $x^{p+3}-3x^{p+2}+x^{p+1}+x^p-x^2+2x+1=0$ and let V_p be a $(p+3)\times(p+3)$ Vandermonde matrix as follows:

$$V_p=\begin{bmatrix}(\alpha_1)^{p+2} & (\alpha_2)^{p+2} & \ldots & (\alpha_{p+3})^{p+2}\\ (\alpha_1)^{p+1} & (\alpha_2)^{p+1} & \ldots & (\alpha_{p+3})^{p+1}\\ \vdots & \vdots & \ddots & \vdots\\ \alpha_1 & \alpha_2 & \ldots & \alpha_{p+3}\\ 1 & 1 & \ldots & 1\end{bmatrix}.$$

Assume that $V_p(i,j)$ is a $(p+3)\times(p+3)$ matrix derived from the Vandermonde matrix V_p by replacing the j-th column of V_p by $W_p(i)$, where, $W_p(i)$ is a $(p+3)\times 1$ matrix as follows:

$$W_p(i)=\begin{bmatrix}(\alpha_1)^{n+p+3-i}\\ (\alpha_2)^{n+p+3-i}\\ \vdots\\ (\alpha_{p+3})^{n+p+3-i}\end{bmatrix}.$$

Theorem 2.4. *Let p be a positive integer such that $p\geq 3$ and let $(D_p)^n=d_{i,j}^{(p,n)}$ for $n\geq 1$, then*

$$d_{i,j}^{(p,n)}=\frac{\det V_p(i,j)}{\det V_p}.$$

Proof. Since the equation $x^{p+3}-3x^{p+2}+x^{p+1}+x^p-x^2+2x+1=0$ does not have multiple roots for $p\geq 3$, the eigenvalues of the Fibonacci–Pell p-matrix D_p are distinct. Then, it is clear that D_p is diagonalizable. Let $A_p=\operatorname{diag}(\alpha_1,\alpha_2,\ldots,\alpha_{p+3})$, then we may write $D_pV_p=V_pA_p$. Since the matrix V_p is invertible, we obtain the equation $(V_p)^{-1}D_pV_p=A_p$. Therefore, D_p is similar to A_p; hence, $(D_p)^nV_p=V_p(A_p)^n$ for $n\geq 1$. So we have the following linear system of equations:

$$\begin{cases} d_{i,1}^{(p,n)}(\alpha_1)^{p+2}+d_{i,2}^{(p,n)}(\alpha_1)^{p+1}+\cdots+d_{i,p+3}^{(p,n)}=(\alpha_1)^{n+p+3-i}\\ d_{i,1}^{(p,n)}(\alpha_2)^{p+2}+d_{i,2}^{(p,n)}(\alpha_2)^{p+1}+\cdots+d_{i,p+3}^{(p,n)}=(\alpha_2)^{n+p+3-i}\\ \vdots\\ d_{i,1}^{(p,n)}(\alpha_{p+3})^{p+2}+d_{i,2}^{(p,n)}(\alpha_{p+3})^{p+1}+\cdots+d_{i,p+3}^{(p,n)}=(\alpha_{p+3})^{n+p+3-i}\end{cases}.$$

Then we conclude that

$$d_{i,j}^{(p,n)}=\frac{\det V_p(i,j)}{\det V_p}$$

for each $i,j=1,2,\ldots,p+3$. □

Thus by Theorem 2.4 and the matrix $(D_p)^n$, we have the following useful result for the Fibonacci–Pell p-numbers.

Corollary 2.1. *Let p be a positive integer such that $p \geq 3$ and let $F_n^{P,p}$ be the n-th element of the Fibonacci–Pell p-sequence, then*

$$F_n^{P,p} = \frac{\det V_p\,(p+3,1)}{\det V_p}$$

and

$$F_n^{P,p} = -\frac{\det V_p\,(p+2,p+3)}{\det V_p}$$

for $n \geq 1$.

It is easy to see that the generating function of the Fibonacci–Pell p-sequence $\left\{F_n^{P,p}\right\}$ is as follows:

$$g(x) = \frac{x^{p+2}}{1-3x+x^2+x^3-x^{p+1}+2x^{p+2}+x^{p+3}},$$

where $p \geq 3$.

Then we can give an exponential representation for the Fibonacci–Pell p-numbers by the aid of the generating function with the following Theorem.

Theorem 2.5. *The Fibonacci–Pell p-numbers $\left\{F_n^{P,p}\right\}$ have the following exponential representation:*

$$g(x) = x^{p+2}\exp\left(\sum_{i=1}^{\infty}\frac{(x)^i}{i}\left(3-x-x^2+x^p-2x^{p+1}-x^{p+2}\right)^i\right),$$

where $p \geq 3$.

Proof. Since

$$\ln g(x) = \ln x^{p+2} - \ln\left(1-3x+x^2+x^3-x^{p+1}+2x^{p+2}+x^{p+3}\right)$$

and

$$\begin{aligned} -\ln\left(1-3x+x^2+x^3-x^{p+1}+2x^{p+2}+x^{p+3}\right) &= -[-x\left(3-x-x^2+x^p-2x^{p+1}-x^{p+2}\right) - \\ &\quad \frac{1}{2}x^2\left(3-x-x^2+x^p-2x^{p+1}-x^{p+2}\right)^2 - \cdots \\ &\quad -\frac{1}{i}x^i\left(3-x-x^2+x^p-2x^{p+1}-x^{p+2}\right)^i - \cdots] \end{aligned}$$

it is clear that

$$g(x) = x^{p+2}\exp\left(\sum_{i=1}^{\infty}\frac{(x)^i}{i}\left(3-x-x^2+x^p-2x^{p+1}-x^{p+2}\right)^i\right)$$

and by a simple calculation, we obtain the conclusion. $\square$

Let $K(k_1, k_2, \ldots, k_v)$ be a $v \times v$ companion matrix as follows:

$$K(k_1, k_2, \ldots, k_v) = \begin{bmatrix} k_1 & k_2 & \cdots & k_v \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Theorem 2.6. (Chen and Louck [3]) *The* (i, j) *entry* $k_{i,j}^{(n)}(k_1, k_2, \ldots, k_v)$ *in the matrix* $K^n(k_1, k_2, \ldots, k_v)$ *is given by the following formula:*

$$k_{i,j}^{(n)}(k_1, k_2, \ldots, k_v) = \sum_{(t_1, t_2, \ldots, t_v)} \frac{t_j + t_{j+1} + \cdots + t_v}{t_1 + t_2 + \cdots + t_v} \times \binom{t_1 + \cdots + t_v}{t_1, \ldots, t_v} k_1^{t_1} \cdots k_v^{t_v}, \tag{2}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + vt_v = n - i + j$, $\binom{t_1+\cdots+t_v}{t_1,\ldots,t_v} = \frac{(t_1+\cdots+t_v)!}{t_1!\cdots t_v!}$ *is a multinomial coefficient, and the coefficients in (2) are defined to be* 1 *if* $n = i - j$.

Then we can give other combinatorial representations than for the Fibonacci–Pell p-numbers by the following Corollary.

Corollary 2.2. *Let* $F_n^{P,p}$ *be the* n*-th Fibonacci–Pell* p*-number for* $n \geq 1$*. Then*

$i.$

$$F_n^{P,p} = \sum_{(t_1, t_2, \ldots, t_{p+3})} \binom{t_1 + t_2 + \cdots + t_{p+3}}{t_1, t_2, \cdots, t_{p+3}} 3^{t_1} (-2)^{t_{p+2}} (-1)^{t_2 + t_3 + t_{p+3}},$$

where the summation is over nonnegative integers satisfying

$$t_1 + 2t_2 + \cdots + (p+3)\, t_{p+3} = n - p - 2.$$

$ii.$

$$F_n^{P,p} = -\sum_{(t_1, t_2, \ldots, t_{p+3})} \frac{t_{p+3}}{t_1 + t_2 + \cdots + t_{p+3}} \times \binom{t_1 + t_2 + \cdots + t_{p+3}}{t_1, t_2, \cdots, t_{p+3}} 3^{t_1} (-2)^{t_{p+2}} (-1)^{t_2 + t_3 + t_{p+3}},$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + (p+3)\, t_{p+3} = n + 1$.

Proof. If we take $i = p + 3$, $j = 1$ for the Case i. and $i = p + 2$, $j = p + 3$ for the Case ii. in Theorem 2.6, then we can directly see the conclusions from $(D_p)^n$. □

Now we consider the relationship between the Fibonacci–Pell p-numbers and the permanent of a certain matrix which is obtained using the Fibonacci–Pell p-matrix $(D_p)^n$.

Definition 2.1. *A* $u \times v$ *real matrix* $M = [m_{i,j}]$ *is called a contractible matrix in the* k*-th column (respectively, row) if the* k*-th column (respectively, row) contains exactly two non-zero entries.*

Suppose that x_1, x_2, …,x_u are row vectors of the matrix M. If M is contractible in the k-th column such that $m_{i,k} \neq 0, m_{j,k} \neq 0$ and $i \neq j$, then the $(u-1) \times (v-1)$ matrix $M_{ij:k}$ obtained from M by replacing the i-th row with $m_{i,k}x_j + m_{j,k}x_i$ and deleting the j-th row. The k-th column is called the contraction in the k-th column relative to the i-th row and the j-th row.

In [2], Brualdi and Gibson obtained that $per\,(M) = per\,(N)$ if M is a real matrix of order $\alpha > 1$ and N is a contraction of M.

Now we concentrate on finding relationships among the Fibonacci–Pell p-numbers and the permanents of certain matrices which are obtained by using the generating matrix of the Fibonacci–Pell p-numbers. Let $E_{m,p}^{F,P} = [e_{i,j}]$ be the $m \times m$ super-diagonal matrix, defined by

$$e_{i,j} = \begin{cases} 3 & \text{if } i = \tau \text{ and } j = \tau \text{ for } 1 \le \tau \le m, \\ 1 & \begin{array}{l}\text{if } i = \tau \text{ and } j = \tau + p \text{ for } 1 \le \tau \le m - p \\ \text{and } \ i = \tau + 1 \text{ and } j = \tau \text{ for } 1 \le \tau \le m - 1,\end{array} \\ -1 & \begin{array}{l}\text{if } i = \tau \text{ and } j = \tau + 1 \text{ for } 1 \le \tau \le m - 1, \\ i = \tau \text{ and } j = \tau + 2 \text{ for } 1 \le \tau \le m - 2 \\ \text{and } \ i = \tau \text{ and } j = \tau + p + 2 \text{ for } 1 \le \tau \le m - p - 2,\end{array} \\ -2 & \text{if } i = \tau \text{ and } j = \tau + p + 1 \text{ for } 1 \le \tau \le m - p - 1, \\ 0 & \text{otherwise.} \end{cases}$$

for $m \ge p + 3$. Then we have the following Theorem.

Theorem 2.7. *For $m \ge p + 3$,*

$$\text{per}\, E_{m,p}^{F,P} = F_{m+p+2}^{P,p}.$$

Proof. Let us consider matrix $E_{m,p}^{F,P}$ and let the equation hold for $m \ge p + 3$. Then we show that the equation holds for $m + 1$. If we expand the per $E_{m,p}^{F,P}$ by the Laplace expansion of permanent with respect to the first row, then we obtain

$$\text{per}\, E_{m+1,p}^{F,P} = 3\,\text{per}\, E_{m,p}^{F,P} - \text{per}\, E_{m-1,p}^{F,P} - \text{per}\, E_{m-2,p}^{F,P} + \text{per}\, E_{m-p,p}^{F,P} - 2\,\text{per}\, E_{m-p-1,p}^{F,P} - \text{per}\, E_{m-p-2,p}^{F,P}.$$

Since

$$\begin{aligned} \text{per}\, E_{m,p}^{F,P} &= F_{m+p+2}^{P,p}, \\ \text{per}\, E_{m-1,p}^{F,P} &= F_{m+p+1}^{P,p}, \\ \text{per}\, E_{m-2,p}^{F,P} &= F_{m+p}^{P,p}, \\ \text{per}\, E_{m-p,p}^{F,P} &= F_{m+2}^{P,p}, \\ \text{per}\, E_{m-p-1,p}^{F,P} &= F_{m+1}^{P,p}, \\ \text{per}\, E_{m-p-2,p}^{F,P} &= F_{m}^{P,p}, \end{aligned}$$

we easily obtain that per $E_{m+1,p}^{F,P} = F_{m+p+3}^{P,p}$. So the proof is complete. □

Let $F^{F,P}_{m,p} = [f_{i,j}]$ be the $m \times m$ matrix, defined by

$$f_{i,j} = \begin{cases} 3 & \text{if } i=\tau \text{ and } j=\tau \text{ for } 1 \le \tau \le m-p, \\ 1 & \begin{array}{l} \text{if } i=\tau \text{ and } j=\tau+p \text{ for } 1 \le \tau \le m-p, \\ i=\tau \text{ and } j=\tau \text{ for } m-p+1 \le \tau \le m, \\ \text{and } \ i=\tau+1 \text{ and } j=\tau \text{ for } 1 \le \tau \le m-p-1, \end{array} \\ -1 & \begin{array}{l} \text{if } i=\tau \text{ and } j=\tau+1 \text{ for } 1 \le \tau \le m-p, \\ i=\tau \text{ and } j=\tau+2 \text{ for } 1 \le \tau \le m-p, \\ \text{and } \ i=\tau \text{ and } j=\tau+p+2 \text{ for } 1 \le \tau \le m-p-2, \end{array} \\ -2 & \text{if } i=\tau \text{ and } j=\tau+p+1 \text{ for } 1 \le \tau \le m-p-1, \\ 0 & \text{otherwise} \end{cases},$$

for $m \ge p+3$. Then we have the following Theorem.

Theorem 2.8. *For $m \ge p+3$,*

$$\operatorname{per} F^{F,P}_{m,p} = F^{P,p}_{m+2}.$$

Proof. Let us consider matrix $F^{F,P}_{m,p}$ and let the equation hold for $m \ge p+3$. Then we show that the equation holds for $m+1$. If we expand the $\operatorname{per} F^{F,P}_{m,p}$ by the Laplace expansion of permanent with respect to the first row, then we obtain

$$\operatorname{per} F^{F,P}_{m+1,p} = 3 \operatorname{per} F^{F,P}_{m,p} - \operatorname{per} F^{F,P}_{m-1,p} - \operatorname{per} F^{F,P}_{m-2,p} + \operatorname{per} F^{F,P}_{m-p,p} - 2 \operatorname{per} F^{F,P}_{m-p-1,p} - \operatorname{per} F^{F,P}_{m-p-2,p}.$$

Since

$$\begin{aligned} \operatorname{per} F^{F,P}_{m,p} &= F^{P,p}_{m+2}, \\ \operatorname{per} F^{F,P}_{m-1,p} &= F^{P,p}_{m+1}, \\ \operatorname{per} F^{F,P}_{m-2,p} &= F^{P,p}_{m}, \\ \operatorname{per} F^{F,P}_{m-p,p} &= F^{P,p}_{m-p+2}, \\ \operatorname{per} F^{F,P}_{m-p-1,p} &= F^{P,p}_{m-p+1}, \\ \operatorname{per} F^{F,P}_{m-p-2,p} &= F^{P,p}_{m-p}, \end{aligned}$$

we easily obtain that $\operatorname{per} F^{F,P}_{m+1,p} = F^{P}_{m+3}$. So the proof is complete. □

Assume that $G^{F,P}_{m,p} = [g_{i,j}]$ be the $m \times m$ matrix, defined by

$$G^{F,P}_{m,p} = \begin{array}{c} \begin{array}{c} (m-3)\text{-rd} \\ \downarrow \end{array} \\ \begin{bmatrix} 1 & \cdots & 1 & 0 & 0 & 0 \\ 1 & & & & & \\ 0 & & & & & \\ \vdots & & & F^{F,P}_{m-1,p} & & \\ 0 & & & & & \\ 0 & & & & & \end{bmatrix} \end{array}, \text{ for } m > p+3,$$

then we have the following results.

Theorem 2.9. *For $m > p + 3$,*

$$\operatorname{per} G^{F,P}_{m,p} = \sum_{i=0}^{m+1} F^{P,p}_i.$$

Proof. If we extend $\operatorname{per} G^{F,P}_{m,p}$ with respect to the first row, we write

$$\operatorname{per} G^{F,P}_{m,p} = \operatorname{per} G^{F,P}_{m-1,p} + \operatorname{per} F^{F,P}_{m-1,p}.$$

Thus, by the results and an inductive argument, the proof is easily seen. $\square$

A matrix M is called convertible if there is an $n \times n$ $(1, -1)$-matrix K such that $\operatorname{per} M = \det (M \circ K)$, where $M \circ K$ denotes the Hadamard product of M and K.

Now we give relationships among the Fibonacci–Pell p-numbers and the determinants of certain matrices which are obtained by using the matrices $E^{F,P}_{m,p}$, $F^{F,P}_{m,p}$ and $G^{F,P}_{m,p}$. Let $m > p + 3$ and let R be the $m \times m$ matrix, defined by

$$R = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & 1 & -1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & -1 & 1 \end{bmatrix}.$$

Corollary 2.3. *For $m > p + 3$,*

$$\det \left(E^{F,P}_{m,p} \circ R \right) = F^{P,p}_{m+p+2},$$

$$\det \left(F^{F,P}_{m,p} \circ R \right) = F^{P,p}_{m+2},$$

and

$$\det \left(G^{F,P}_{m,p} \circ R \right) = \sum_{i=0}^{m+1} F^{P,p}_i.$$

Proof. Since $\operatorname{per} E^{F,P}_{m,p} = \det \left(E^{F,P}_{m,p} \circ R \right)$, $\operatorname{per} F^{F,P}_{m,p} = \det \left(F^{F,P}_{m,p} \circ R \right)$ and $\operatorname{per} G^{F,P}_{m,p} = \det \left(G^{F,P}_{m,p} \circ R \right)$ for $m > p + 3$, by Theorem 2.7, Theorem 2.8 and Theorem 2.9, we have the conclusion. $\square$

Now we consider the sums of the Fibonacci–Pell p-numbers. Let

$$S_n = \sum_{i=0}^{n} F^P_i$$

for $n \geq 0$ and let $U_{F,P}$ and $(U_{F,P})^n$ be the $(p + 4) \times (p + 4)$ matrix such that

$$U_{F,P} = \left[\begin{array}{c|ccccc} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & & & & & \\ 0 & & & & & \\ \vdots & & & D_p & & \\ 0 & & & & & \\ 0 & & & & & \end{array} \right].$$

If we use induction on n, then we obtain

$$(U_{F,P})^n = \begin{bmatrix} 1 & 0 \;\; 0 \;\; \cdots \;\; 0 \;\; 0 \\ S_{n+p+1} & \\ S_{n+p4} & \\ \vdots & (D_p)^n \\ S_n & \\ S_{n-1} & \end{bmatrix}.$$

References

[1] Bradie, B. (2010). Extension and refinements of some properties of sums involving Pell number. *Missouri Journal of Mathematical Sciences*, 22(1), 37–43.

[2] Brualdi, R. A., & Gibson, P. M. (1977). Convex polyhedra of doubly stochastic matrices. I. Applications of permanent function, *Journal of Combinatorial Theory, Series A*, 22(2), 194–230.

[3] Chen, W. Y. C., & Louck, J. D. (1996). The combinatorial power of the companion matrix. *Linear Algebra and its Applications*, 232, 261–278.

[4] Devaney, R. L. (1999). The Mandelbrot set, the Farey tree, and the Fibonacci sequence, *The American Mathematical Monthly*, 106(4), 289–302.

[5] Deveci, O., Adiguzel, Z. & Dogan, T. (2020). On the Generalized Fibonacci-circulant-Hurwitz Numbers, *Notes on Number Theory and Discrete Mathematics*, 26(1), 179–190.

[6] Deveci, O., & Artun, G. (2019). On the Adjacency-Jacobsthal Numbers. *Communications in Algebra*, 47 (11), 4520-4532.

[7] Deveci, O., Karaduman, E., & Campbell, C. M. (2017). The Fibonacci-Circulant Sequences and Their Applications. *Iranian Journal of Science and Technology, Transaction A, Science*, 41(4), 1033–1038.

[8] Frey, D. D., & Sellers, J. A. (2000). Jacobsthal numbers and alternating sign matrices. *Journal of Integer Sequences*, 3, Article 00.2.3.

[9] Gogin, N. D., & Myllari, A. A. (2007). The Fibonacci–Padovan sequence and MacWilliams transform matrices. *Program. Comput. Softw., published in Programmirovanie*, 33(2), 74–79.

[10] Horadam, A. F. (1994). Applications of modified Pell numbers to representations. *Ulam Quarterly Journal*, 3(1), 34–53.

[11] Johnson, R. C. (2009). *Fibonacci numbers and matrices*, Available online: `https://maths.dur.ac.uk/~dma0rcj/PED/fib.pdf`.

[12] Kalman, D. (1982). Generalized Fibonacci numbers by matrix methods. *The Fibonacci Quarterly*, 20(1), 73–76.

[13] Kilic, E. (2008). The Binet fomula, sums and representations of generalized Fibonacci p-numbers. *European Journal of Combinatorics*, 29(3), 701–711.

[14] Kilic, E., & Tasci, D. (2006). The generalized Binet formula, representation and sums of the generalized order-k Pell numbers. *Taiwanese Journal of Mathematics*, 10(6), 1661–1670.

[15] Kocer, E. G., & Tuglu, N. (2007). The Binet formulas for the Pell and Pell–Lucas p-numbers. *Ars Combinatoria*, 85, 3–17.

[16] Koken, F., & Bozkurt, D. (2008). On the Jacobsthal numbers by matrix methods. *International Journal of Contemporary Mathematical Sciences*, 3(13), 605–614.

[17] Lancaster, P. & Tismenetsky, M. (1985). *The Theory of Matrices: with Applications*, Elsevier.

[18] Lidl, R., & Niederreiter, H. (1994). *Introduction to Finite Fields and Their Applications*, Cambridge University Press.

[19] McDaniel, W. L. (1996). Triangular numbers in the Pell sequence. *The Fibonacci Quarterly*, 34(2), 105–107.

[20] Shannon, A. G., Anderson, P. G., & Horadam, A. F. (2006). Properties of Cordonnier, Perrin and Van der Laan numbers. *International Journal of Mathematical Education in Science and Technology*, 37(7), 825–831.

[21] Shannon, A. G., Horadam, A. F., & Anderson, P. G. (2006). The auxiliary equation associated with the plastic number, *Notes on Number Theory and Discrete Mathematics*, 12(1), 1–12.

[22] Stakhov, A. P. (1999). A generalization of the Fibonacci Q-matrix. *Rep. Natl. Acad. Sci. Ukraine*, 9, 46–49.

[23] Stakhov, A. P., & Rozin, B. (2006). Theory of Binet formulas for Fibonacci and Lucas p-numbers. *Chaos, Solitions, Fractals*, 27(5), 1162–1177.

[24] Stewart, I. (1996). Tales of a neglected number. *Scientific American*, 274(6), 102–103.

[25] Tasci, D., & Firengiz, M. C. (2010). Incomplete Fibonacci and Lucas p-numbers. *Mathematical and Computer Modelling*, 52(9–10), 1763–1770.