

Characterization of ideals of rhotrices over a ring and its applications

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Received: 21 December 2019 **Revised:** 17 September 2020 **Accepted:** 28 January 2021

Abstract: We define higher order rhotrices over a commutative unital ring S and obtain a ring $\mathcal{R}_n(S)$ of rhotrices of the order $n \in \mathbb{N}$. We characterize the ideals and maximal ideals of $\mathcal{R}_n(S)$. As a particular case, we record ideals of rhotrix rings over integers and rhotrix algebras over complex plane \mathbb{C} . As an application, we characterize the maximal ideals of the commutative unital Banach algebra $\mathcal{R}_n(\mathbb{C})$.

Keywords: Rhotrix over a ring, Unital ring, Maximal ideal, Banach algebra.

2010 Mathematics Subject Classification: 15B99, 46J20.

1 Introduction

The concept of rhotrices was introduced by Ajibade [1] in 2003 as objects of order higher than 2×2 and lower than 3×3 matrices, as an extension of the initiative on *matrix-tertions* and *matrix-noitrets* suggested by Atanassov and Shannon [2]. In literature it is realized that with a slight perturbation of the arrangement of a matrix, a rhotrix – a rhomboidal array is obtained. As in [1], a *rhotrix* of order 3 is defined as

$$R = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle,$$

where $a, b, d, e, h(R) \in \mathbb{R}$. $h(R)$ is called the *heart of the rhotrix*. For higher order rhotrices, it is difficult to realize a rhotrix as objects of order higher than $(n-1) \times (n-1)$ and lower than $n \times n$

matrices. But it will be convenient to see it as a part of $n \times n$ matrix. Thus a rhotrix of order 3 is looked upon as a part of 3×3 matrix

$$A_R = \begin{bmatrix} a & * & d \\ * & h(R) & * \\ b & * & e \end{bmatrix}$$

rotated through $h(R)$ clockwise by 45° and removing $*$. The concept was generalized to a rhotrix of order n by Mohammed [7]. He defined a rhotrix R of order n as

$$R = \left(\begin{array}{cccccccc} & & & & a_1 & & & & \\ & & & & a_2 & a_3 & a_4 & & & \\ & & \dots & \dots & \dots & \dots & \dots & & & \\ a_{\frac{t-n+2}{2}} & \dots & \dots & a_{\frac{t+1}{2}} & \dots & \dots & \dots & a_{\frac{t+n}{2}} & & \\ & & \dots & \dots & \dots & \dots & \dots & & & \\ & & & & a_{t-3} & a_{t-2} & a_{t-1} & & & \\ & & & & & & a_t & & & \end{array} \right), \quad (1)$$

where $t = \frac{n^2+1}{2}$ and $a_i \in \mathbb{R}$. Here $h(R) = a_{\frac{t+1}{2}}$ is called the *heart of R*.

We shall denote the rhotrix defined in (1) by $R = \langle h(R), a_i \rangle$, $1 \leq i \leq t$, $i \neq \frac{t+1}{2}$. In what follows, for brevity, when $h(R)$ is mentioned separately, we shall assume tacitly that $1 \leq i \leq t$ and $i \neq \frac{t+1}{2}$.

The set of all rhotrices of order n is denoted by $\mathcal{R}_n(\mathbb{R})$. The heart based product on $\mathcal{R}_n(\mathbb{R})$ is defined in [7] as shown below. Let $R_1 = \langle h(R_1), a_i \rangle$, $R_2 = \langle h(R_2), b_i \rangle \in \mathcal{R}_n(\mathbb{R})$, $1 \leq i \leq t$. Then

$$R_1 R_2 = \langle h(R_1)h(R_2), h(R_1)b_i + h(R_2)a_i \rangle.$$

In passing we note that for a fixed $k \neq \frac{t+1}{2}$ the two dimensional subspace $V = \{(h(R), a_k) : R \in \mathcal{R}_n(\mathbb{R})\}$ becomes a Banach algebra. This and several other multiplications on a two dimensional Banach algebra are investigated in [3].

On the other hand, motivated by matrix multiplication, Sani [10] multiplied two rhotrices of order n in a different way. To realize this we write a rhotrix of order n as a fusion of two matrices, one matrix $[a_{ij}]$ of the order $\frac{n+1}{2}$ and the other $[c_{ij}]$ of the order $\frac{n-1}{2}$. Thus a rhotrix of order n , written differently, is

$$R = \langle a_{ij}, c_{lk} \rangle = \left(\begin{array}{cccccccc} & & & & a_{11} & & & & \\ & & & & a_{21} & c_{11} & a_{12} & & & \\ & & \dots & \dots & \dots & \dots & \dots & & & \\ a_{t1} & c_{(t-1)1} & \dots & \dots & \dots & \dots & \dots & c_{1(t-1)} & a_{1t} & \\ & & \dots & \dots & \dots & \dots & \dots & & & \\ & & & & a_{t(t-1)} & c_{(t-1)(t-1)} & a_{(t-1)t} & & & \\ & & & & & & a_{tt} & & & \end{array} \right),$$

where $t = \frac{n+1}{2}$ and the number of components of R is given by $|R| = \frac{n^2+1}{2}$. For two rhotrices $R_1 = \langle a_{ij}, c_{lk} \rangle$, $R_2 = \langle b_{ij}, d_{lk} \rangle$, the (noncommutative) product is defined as

$$R_1 \circ R_2 = \left\langle \sum_{q=1}^t a_{iq} b_{qj}, \sum_{q=1}^{t-1} c_{lq} d_{qk} \right\rangle$$

Mohammed and Okon [8] investigated the noncommutative group structure of general rhotrices of order n with noncommutative product. We consider the commutative product and investigate ring theoretic and algebra theoretic structures of rhotrices by allowing the elements of rhotrices to come from a ring, the set of integers or complex plane. In yet another direction, Isere [4] and [5] defined heartless rhotrices incorporating the even ordered rhotrices. But in the present note, our arrangement is possible only when n is odd. Thus throughout this note, when we refer to n as an *order of a rhotrix*, n will be assumed to be a positive odd integer.

As mentioned in the above definition of the rhotrix R , it is a fusion of two matrices $\langle a_{ij}, c_{lk} \rangle$, which form a part of an $n \times n$ matrix

$$A_R = \begin{bmatrix} a_{11} & * & \cdots & \cdots & \cdots & * & a_{1t} \\ * & c_{11} & \cdots & * & \cdots & c_{1(t-1)} & * \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & * & \cdots & a_{\frac{t+1}{2} \frac{t+1}{2}} & \cdots & * & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ * & c_{(t-1)(1)} & \cdots & * & \cdots & c_{(t-1)(t-1)} & * \\ a_{t1} & * & \cdots & \cdots & \cdots & * & a_{tt} \end{bmatrix}.$$

In this case, $[a_{ij}], [c_{lk}]$ are called the *coupled matrices* Sani [11]. Various concepts related to rhotrices are investigated by different researchers. To mention a few, Patil, Singh and Sutaria [9] investigated eigenvalues and eigenvectors of a rhotrix corresponding to a matrix, Sharma and Kanwar [12] have proved Cayley–Hamilton theorem for rhotrices. Usaini and Aminu [13] have defined and investigated the exponential functions on rhotrices.

Ajibade [1] proved that under the heart based multiplication, rhotrices form a semigroup. He also obtained the necessary and sufficient condition for a rhotrix to be invertible. Mohammed [6] investigates the ring theoretic structure of the same set obtaining ring theoretic results. Both of them restricted themselves to $\mathcal{R}_3(\mathbb{R})$, the rhotrices of order 3. In this paper, we consider rhotrices of order n with entries from a commutative unital ring S . Throughout this paper S will denote a commutative unital ring. We denote the identity of S by $\mathbf{1}$.

Definition 1.1. By a rhotrix of order n over S , we mean a rhotrix

$$R = \left\langle \begin{array}{cccccccc} & & & & a_1 & & & \\ & & & & a_2 & a_3 & a_4 & \\ & & & & \cdots & \cdots & \cdots & \\ a_{\frac{t-n+2}{2}} & \cdots & \cdots & a_{\frac{t+1}{2}} & \cdots & \cdots & a_{\frac{t+n}{2}} & \\ & & & \cdots & \cdots & \cdots & \cdots & \\ & & & a_{t-3} & a_{t-2} & a_{t-1} & & \\ & & & & a_t & & & \end{array} \right\rangle,$$

where $t = \frac{n^2 + 1}{2}$ and $a_i \in S$. Here $h(R) = a_{\frac{t+1}{2}}$ is the *heart* of R .

Throughout this note $\mathcal{R}_n(S)$ will denote the set of all rhotrices of order n over S . Let $R_1 = \langle h(R_1), a_i \rangle$ and $R_2 = \langle h(R_2), b_i \rangle$ be in $\mathcal{R}_n(S)$. We define

$$\begin{aligned} R_1 + R_2 &= \langle h(R_1) + h(R_2), a_i + b_i \rangle, \\ R_1 R_2 &= \langle h(R_1)h(R_2), h(R_1)b_i + h(R_2)a_i \rangle. \end{aligned} \quad (2)$$

In the main result Theorem 2.2, in Section 2, we characterize ideals of the ring $\mathcal{R}_n(S)$ of rhotrices over a commutative unital ring S . Some very interesting particular cases of rhotrix rings and rhotrix algebras are recorded. As an application of Theorem 2.2, in Section 3, we define Rhotrix Banach algebra and find its maximal ideals and multiplicative linear functionals.

2 Ideals of $\mathcal{R}_n(S)$

The following is a straight verification using (2).

Proposition 2.1. $\mathcal{R}_n(S)$ is a commutative unital ring.

Theorem 2.2. $\mathcal{I} \subset \mathcal{R}_n(S)$ is an ideal of $\mathcal{R}_n(S)$ if and only if there is a family $\{\mathcal{I}_i : 1 \leq i \leq t\}$ of ideals of S such that $\mathcal{J}_h = \mathcal{I}_{\frac{t+1}{2}} \subset \mathcal{I}_i$ for every i and

$$\mathcal{I} = \{\langle h(R), a_i \rangle : a_i \in \mathcal{I}_i, h(R) \in \mathcal{J}_h\}. \quad (3)$$

Proof. Suppose there is a family $\{\mathcal{I}_i : 1 \leq i \leq t\}$ of ideals of S such that $\mathcal{J}_h \subset \mathcal{I}_i$ for every i and \mathcal{I} is given by (3). We show that \mathcal{I} is an ideal of $\mathcal{R}_n(S)$. Since \mathcal{I}_i are ideals of S and since the addition on $\mathcal{R}_n(S)$ is pointwise, it is clear that \mathcal{I} is an additive subgroup of $\mathcal{R}_n(S)$. Let $R_1 = \langle h(R_1), a_i \rangle \in \mathcal{R}_n(S)$, $R_2 = \langle h(R_2), b_i \rangle \in \mathcal{I}$. Then for every i , $h(R_1)b_i \in S\mathcal{I}_i \subset \mathcal{I}_i$ because \mathcal{I}_i is an ideal of S . Also, using the fact that $\mathcal{J}_h \subset \mathcal{I}_i$, we have $h(R_2)a_i \in \mathcal{J}_h S \subset \mathcal{I}_i S \subset \mathcal{I}_i$. Thus $h(R_1)b_i + h(R_2)a_i \in \mathcal{I}_i$. Also, $h(R_1)h(R_2) \in S\mathcal{J}_h \subset \mathcal{J}_h$ because \mathcal{J}_h is an ideal of S . Thus $R_1 R_2 \in \mathcal{I}$.

Conversely, suppose \mathcal{I} is an ideal of $\mathcal{R}_n(S)$. We fix k for the rest of the proof. Define

$$\mathcal{J}_h = \{h(R) : R \in \mathcal{I}\}$$

and

$$\mathcal{I}_k = \{a_k : R = \langle h(R), a_i \rangle \in \mathcal{I}\}.$$

First we show that \mathcal{J}_h is an ideal of S . Let $c, d \in \mathcal{J}_h$. So, there are $R_1 = \langle h(R_1), a_i \rangle, R_2 = \langle h(R_2), b_i \rangle \in \mathcal{I}$ such that $h(R_1) = c$ and $h(R_2) = d$. Then $c + d = h(R_1) + h(R_2) = h(R_1 + R_2)$. Since $R_1 + R_2 \in \mathcal{I}$, it follows that $c + d \in \mathcal{J}_h$. Let $c \in \mathcal{J}_h$ and $d \in S$. So, there is $R_1 = \langle h(R_1), a_i \rangle \in \mathcal{I}$ such that $h(R_1) = c$. Define $R_2 = \langle h(R_2), b_i \rangle \in \mathcal{R}_n(S)$ by setting $h(R_2) = d$ and $b_i = 0$ for every i . Then $cd = h(R_1)h(R_2) = h(R_1 R_2)$. Since \mathcal{I} is an ideal of $\mathcal{R}_n(S)$, $R_1 R_2 \in \mathcal{I}$. Hence $cd \in \mathcal{J}_h$. Thus \mathcal{J}_h is an ideal of S . Next we show that \mathcal{I}_k is an ideal of S . Let $c, d \in \mathcal{I}_k$. So, there are $R_1 = \langle h(R_1), a_i \rangle, R_2 = \langle h(R_2), b_i \rangle \in \mathcal{I}$ such that $a_k = c$ and $b_k = d$.

Since \mathcal{I} is an ideal, $R_1 + R_2 \in \mathcal{I}$. Hence it follows that $c + d = a_k + b_k \in \mathcal{I}_k$. Let $c \in \mathcal{I}_k$ and $d \in S$. So, there is $R_1 = \langle h(R_1), a_i \rangle \in \mathcal{I}$ such that $a_k = c$. Define $R_2 = \langle h(R_2), b_i \rangle \in \mathcal{R}_n(S)$

by setting $h(R_2) = d$ and $b_i = 0$ for every i . Now the k -th coordinate of R_1R_2 is $h(R_1)b_k + h(R_2)a_k = h(R_1)0 + dc = cd$. Since \mathcal{I} is an ideal of $\mathcal{R}_n(S)$, $R_1R_2 \in \mathcal{I}$ giving $cd \in \mathcal{I}_k$. Thus \mathcal{I}_k is an ideal of S . Finally, we show that $\mathcal{J}_h \subset \mathcal{I}_k$. Let $c \in \mathcal{J}_h$. So, there is $R_1 = \langle h(R_1), a_i \rangle \in \mathcal{I}$ such that $h(R_1) = c$. Define $R_2 = \langle h(R_2), b_i \rangle \in \mathcal{R}_n(S)$ such that $h(R_2) = 0$ and $b_i = 1$ for every i . Now the k -th coordinate of R_1R_2 is $h(R_1)b_k + h(R_2)a_k = c \cdot 1 + 0 \cdot a_k = c$. Since \mathcal{I} is an ideal of $\mathcal{R}_n(S)$, $R_1R_2 \in \mathcal{I}$, which gives $c \in \mathcal{I}_k$. \square

We cannot drop the condition that S is unital. The following example shows that it is necessary.

Example 2.3. Consider $\mathcal{R}_3(2\mathbb{Z})$ and

$$\mathcal{I} = \left\{ \left\langle \begin{array}{ccc} a & & \\ b & h(R) & d \\ & e & \end{array} \right\rangle : a, b, d, e \in 4\mathbb{Z}, h(R) \in 2\mathbb{Z} \right\}.$$

Then one easily verifies that \mathcal{I} is an ideal of $\mathcal{R}_3(2\mathbb{Z})$, however \mathcal{J}_h is not a subset of \mathcal{I}_1 .

As an application of the Theorem 2.2, we obtain the characterization of maximal ideals of $\mathcal{R}_n(S)$.

Theorem 2.4. An ideal \mathcal{I} of $\mathcal{R}_n(S)$ is maximal if and only if \mathcal{J}_h is a maximal ideal of S and $\mathcal{I}_i = S$ for every $i \neq \frac{t+1}{2}$.

Proof. Suppose $\mathcal{I}_k \subsetneq S$. Then we define

$$\mathcal{K} = \{ \langle h(R), a_i \rangle : a_i \in \mathcal{I}_i, \text{ for every } i \neq k, h(R) \in \mathcal{J}_h, a_k \in S \}.$$

Then $\mathcal{I} \subsetneq \mathcal{K}$. Thus it follows that $\mathcal{I}_k = S$. Now suppose \mathcal{J}_h is not maximal in S . So that there is an ideal \mathcal{K}_h of S such that $\mathcal{J}_h \subsetneq \mathcal{K}_h$. Thus allowing the elements of heart to come from \mathcal{K}_h instead of \mathcal{J}_h as above, we see that maximality of \mathcal{I} is annihilated. Thus the result follows. \square

In the rest of this section, we give some particular cases of rhotrix rings. In fact, the results are true for rhotrices of order n for any $n \in \mathbb{N}$, but to save the space, we illustrate the same for rhotrices of order 3. For brevity, first we change the notations in this section. The set

$$\mathcal{I} = \left\{ \left\langle \begin{array}{ccc} a_1 & & \\ a_2 & h(R) & a_4 \\ & a_5 & \end{array} \right\rangle : a_i \in m_i\mathbb{Z}, \text{ for } i = 1, 2, 4, 5, h(R) \in m_3\mathbb{Z} \right\}$$

is written as $\left\langle \begin{array}{ccc} m_1\mathbb{Z} & & \\ m_2\mathbb{Z} & m_3\mathbb{Z} & m_4\mathbb{Z} \\ & m_5\mathbb{Z} & \end{array} \right\rangle$.

Also, given $\mathcal{I} \subset \mathcal{R}_3(\mathbb{Z})$ and $k = 1, 2, 4, 5$, we define

$$\mathcal{I}_k = \{ a_k : R = \langle h(R), a_i \rangle \in \mathcal{I} \}$$

and

$$\mathcal{J}_h = \{ h(R) : R \in \mathcal{I} \}.$$

Corollary 2.4.1. \mathcal{I} is an ideal of $R_3(\mathbb{Z})$ if and only if

$$\mathcal{I} = \left\langle \begin{array}{ccc} & m_1\mathbb{Z} & \\ m_2\mathbb{Z} & m_3\mathbb{Z} & m_4\mathbb{Z} \\ & m_5\mathbb{Z} & \end{array} \right\rangle$$

for some integers m_1, m_2, \dots, m_5 and $m_k | m_3$ for every k .

Proof. Though the proof follows from the Theorem 2.2, we give the direct proof. One way proof follows by noting in the following product that $m_i | (a_3 m_i b_i + m_3 b_3 a_i)$ because $m_i | m_3$, for

every $i = 1, 2, 4, 5$ and $a_1, \dots, a_5, b_1, \dots, b_5 \in \mathbb{Z}$, where $A = \left\langle \begin{array}{ccc} & a_1 & \\ a_2 & a_3 & a_4 \\ & a_5 & \end{array} \right\rangle \in R_3(\mathbb{Z})$ and

$$B = \left\langle \begin{array}{ccc} & m_1 b_1 & \\ m_2 b_2 & m_3 b_3 & m_4 b_4 \\ & m_5 b_5 & \end{array} \right\rangle \in \mathcal{I}.$$

$$\begin{aligned} & \left\langle \begin{array}{ccc} & a_1 & \\ a_2 & a_3 & a_4 \\ & a_5 & \end{array} \right\rangle \left\langle \begin{array}{ccc} & m_1 b_1 & \\ m_2 b_2 & m_3 b_3 & m_4 b_4 \\ & m_5 b_5 & \end{array} \right\rangle \\ &= \left\langle \begin{array}{ccc} & a_3 m_1 b_1 + m_3 b_3 a_1 & \\ a_3 m_2 b_2 + m_3 b_3 a_2 & a_3 m_3 b_3 & a_3 m_4 b_4 + m_3 b_3 a_4 \\ & a_3 m_5 b_5 + a_5 m_3 b_3 & \end{array} \right\rangle. \end{aligned}$$

Conversely, suppose \mathcal{I} is an ideal of $R_3(\mathbb{Z})$. Following the notations defined before the statement, first we show that \mathcal{J}_h as well as \mathcal{I}_k ($k = 1, 2, 4, 5$) is an ideal of \mathbb{Z} . We fix $k \in \{1, 2, 4, 5\}$. Let $a, b \in \mathcal{I}_k$. So, there are $R_1 = \langle h(R_1), a_i \rangle \in \mathcal{I}, R_2 = \langle h(R_2), b_i \rangle \in \mathcal{I}$ such that $a = a_k$, and $b = b_k$. Hence $a - b = a_k - b_k \in \mathcal{I}_k$ because $R_1 - R_2 = \langle h(R_1 - R_2), a_i - b_i \rangle \in \mathcal{I}$. Thus \mathcal{I}_k is a subgroup of \mathbb{Z} . Similarly, \mathcal{J}_h is also a subgroup of \mathbb{Z} . Hence from the basic result of the algebra and number theory, there are $m_i \in \mathbb{Z}, (1 \leq i \leq 5)$ such that $\mathcal{I}_k = m_k \mathbb{Z}$ and $\mathcal{J}_h = m_3 \mathbb{Z}$.

Thus we conclude that the ideal \mathcal{I} must be of the form $\mathcal{I} = \left\langle \begin{array}{ccc} & m_1\mathbb{Z} & \\ m_2\mathbb{Z} & m_3\mathbb{Z} & m_4\mathbb{Z} \\ & m_5\mathbb{Z} & \end{array} \right\rangle$. To settle the

final claim, we note that $\left\langle \begin{array}{ccc} & m_3 & \\ m_3 & m_3 & m_3 \\ & m_3 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 1 & \\ 1 & 1 & 1 \\ & 1 & \end{array} \right\rangle \left\langle \begin{array}{ccc} & 0 & \\ 0 & m_3 & 0 \\ & 0 & \end{array} \right\rangle \in \mathcal{I}$. This shows that

$m_k | m_3$ for every k . □

Corollary 2.4.2.

$$\mathcal{I} = \left\langle \begin{array}{ccc} & m_1\mathbb{Z} & \\ m_2\mathbb{Z} & m_3\mathbb{Z} & m_4\mathbb{Z} \\ & m_5\mathbb{Z} & \end{array} \right\rangle$$

with $m_1, m_2, \dots, m_5 \in \mathbb{Z}$ is a maximal ideal of $\mathcal{R}_3(\mathbb{Z})$ if and only if $m_1, m_2, m_4, m_5 \in \{-1, 1\}$ and m_3 is prime.

Proof. One way it is clear that if $m_1, m_2, m_4, m_5 \in \{-1, 1\}$ and m_3 is prime, then \mathcal{I} is a maximal ideal. Conversely, if $m_i \notin \{\pm 1\}$ for some $i = 1, 2, 4, 5$, then $\mathcal{I} \subsetneq \left\langle \begin{array}{ccc} \mathbb{Z} & & \\ \mathbb{Z} & m_3\mathbb{Z} & \\ & \mathbb{Z} & \mathbb{Z} \end{array} \right\rangle \subsetneq R_3(\mathbb{Z})$. This contradiction shows that $m_1, m_2, m_4, m_5 \in \{-1, 1\}$. Also if m_3 is not prime, then choosing a prime factor p of m_3 , we see that $\mathcal{I} \subsetneq \left\langle \begin{array}{ccc} \mathbb{Z} & & \\ \mathbb{Z} & p\mathbb{Z} & \\ & \mathbb{Z} & \mathbb{Z} \end{array} \right\rangle \subsetneq R_3(\mathbb{Z})$. This confirms that m_3 must be prime. This completes the proof. \square

The following deals with the rhotrix algebra $\mathcal{R}_n(\mathbb{F})$, where \mathbb{F} is a field.

Corollary 2.4.3.

$$\mathcal{I} = \left\langle \begin{array}{ccc} \mathcal{I}_1 & & \\ \mathcal{I}_2 & \mathcal{I}_3 & \mathcal{I}_4 \\ & & \mathcal{I}_5 \end{array} \right\rangle$$

is an ideal of $\mathcal{R}_3(\mathbb{F})$ if and only if for every $1 \leq i \leq 5$ \mathcal{I}_i is either $\{0\}$ or \mathbb{F} . Further, \mathcal{I} is a maximal ideal if and only if $\mathcal{I}_i = \mathbb{F}$ for every $i \neq 3$ and $\mathcal{I}_3 = \{0\}$.

Proof. Arguing in the same way as in the Corollary 2.4.2 and replacing \mathbb{Z} by \mathbb{F} , we see that \mathcal{I}_k are the ideals of \mathbb{F} for every k . Since the ideals of a field \mathbb{F} are $\{0\}$ and \mathbb{F} only, the proof follows. \square

3 Rhotrix Banach algebras

We recall that a *Banach algebra* is a vector space \mathcal{A} over \mathbb{R} or \mathbb{C} endowed with a complete norm $\|\cdot\|$ and a ring multiplication $(x, y) \in \mathcal{A} \times \mathcal{A} \mapsto xy \in \mathcal{A}$ satisfying, for all $x, y, z \in \mathcal{R}_n(\mathbb{C})$,

$$\begin{aligned} x(y + z) &= xy + xz \\ (y + z)x &= yx + zx \\ \alpha(xy) &= (\alpha x)y = x(\alpha y) \\ \|xy\| &\leq \|x\|\|y\| \\ \|\mathbf{1}\| &= 1. \end{aligned}$$

We illustrate the concept of Banach algebras by means putting on record some standard Banach algebras.

Example 3.1. 1. \mathbb{C}^n with the coordinatewise operations and the following norm is a commutative unital Banach algebra.

$$\|(z_1, z_2, z_3, \dots, z_n)\|_\infty = \max\{|z_i| : 1 \leq i \leq n\}, (z_1, z_2, z_3, \dots, z_n) \in \mathbb{C}^n.$$

2. Function Algebras

(a) For a compact Hausdorff topological space X , let $C(X) = \{f : X \rightarrow \mathbb{C} : f \text{ is continuous on } X\}$. Then with the pointwise operations and the following norm (as known as supnorm) $C(X)$ is a commutative unital Banach algebra.

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}, (f \in C(X)).$$

(b) Let $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ and $A(\mathbb{D}) = \{f : \mathbb{D} \rightarrow \mathbb{C} : f \text{ is analytic on the interior of } \mathbb{D} \text{ and continuous on the boundary of } \mathbb{D}\}$. Then $A(\mathbb{D})$ is a Banach algebra with identity with the supnorm defined as follows.

$$\|f\|_\infty = \sup\{|f(z)| : z \in \mathbb{D}\}, (f \in A(\mathbb{D})).$$

There are well known Banach algebras of operators on Hilbert spaces, sequence algebras and algebras of differentiable functions on compact manifolds.

The present note gives a new class of Banach algebras, known as *rotrix algebras*, that have not yet appeared anywhere in the literature so far.

It is easy to see that $\mathcal{R}_n(\mathbb{C})$ is a vector space as well as a ring with heart based multiplication. Note that the identity of $\mathcal{R}_n(\mathbb{C})$ is $\mathbf{1} = \langle h(\mathbf{1}), a_i \rangle$, where $h(\mathbf{1}) = 1$ and $a_i = 0$ for every i .

Definition 3.2. Let $n \in \mathbb{N}$. For $\langle h(R), a_i \rangle \in \mathcal{R}_n(\mathbb{C})$, we define the norm

$$\|\langle h(R), a_i \rangle\| = |h(R)| + \max_i |a_i|. \quad (4)$$

Lemma 3.3. $\mathcal{R}_n(\mathbb{C})$ is a Banach algebra with the norm defined in (4).

Proof. Clearly, $\|\cdot\|$ is a vector space norm on $\mathcal{R}_n(\mathbb{C})$, i.e., for $R_1, R_2 \in \mathcal{R}_n(\mathbb{C})$, and $\alpha \in \mathbb{C}$,

$$\begin{aligned} \|R_1\| &\geq 0 \text{ and } \|R_1\| = 0 \text{ iff } R_1 = 0 \\ \|R_1 + R_2\| &\leq \|R_1\| + \|R_2\| \\ \|\alpha R_1\| &= |\alpha| \|R_1\|. \end{aligned}$$

To prove the submultiplicativity of the norm, let $R_1 = \langle h(R_1), a_i \rangle, R_2 = \langle h(R_2), b_i \rangle \in \mathcal{R}_n(\mathbb{C})$. Then

$$\begin{aligned} \|R_1 R_2\| &= |h(R_1)h(R_2)| + \max_i \{|h(R_1)b_i + h(R_2)a_i|\} \\ &\leq |h(R_1)||h(R_2)| + \max_i \{|h(R_1)b_i| + |h(R_2)a_i|\} \\ &\leq |h(R_1)||h(R_2)| + \max_i \{|h(R_1)b_i|\} + \max_i \{|h(R_2)a_i|\} \\ &= |h(R_1)||h(R_2)| + |h(R_1)| \max_i \{|b_i|\} + |h(R_2)| \max_i \{|a_i|\} \\ &\leq |h(R_1)||h(R_2)| + |h(R_1)| \max_i \{|b_i|\} + |h(R_2)| \max_i \{|a_i|\} \\ &\quad + \max_i \{|a_i|\} \max_i \{|b_i|\} \\ &= |h(R_1)|(|h(R_2)| + \max_i \{|b_i|\}) + \max_i \{|a_i|\}(|h(R_2)| + \max_i \{|b_i|\}) \\ &= (|h(R_1)| + \max_i \{|a_i|\})(|h(R_2)| + \max_i \{|b_i|\}) \\ &= \|R_1\| \|R_2\|. \end{aligned}$$

This completes the proof. □

We demonstrate the computations in the following example.

Example 3.4. Let $R_1 = \left\langle \begin{array}{ccc} & 1 & \\ 1 & 2 & 1 \\ & & 1 \end{array} \right\rangle$ and $R_2 = \left\langle \begin{array}{ccc} & 5 & \\ 1 & 1 & 1 \\ & & 1 \end{array} \right\rangle$. Then

$$\begin{aligned} R_1 R_2 &= \left\langle \begin{array}{ccc} & 2 \times 5 + 1 \times 1 & \\ 2 \times 1 + 1 \times 1 & 2 \times 1 & 2 \times 1 + 1 \times 1 \\ & 2 \times 1 + 1 \times 1 & \end{array} \right\rangle \\ &= \left\langle \begin{array}{ccc} & 11 & \\ 3 & 2 & 3 \\ & & 3 \end{array} \right\rangle. \end{aligned}$$

Also,

$$\begin{aligned} \|R_1\| &= |2| + \max\{1, 1, 1, 1\} = 3 \\ \|R_2\| &= |1| + \max\{5, 1, 1, 1\} = 6 \\ \|R_1 R_2\| &= |2| + \max\{11, 3, 3, 3\} = 13. \end{aligned}$$

Hence,

$$\|R_1 R_2\| \leq \|R_1\| \|R_2\|.$$

Remark 3.5. The most natural supnorm is not carried over to $\mathcal{R}_n(\mathbb{C})$ because of its heart based multiplication. In fact the submultiplicativity of supnorm fails as can be seen from the following.

For rhotrix $R_1 = \langle h(R_1), a_i \rangle \in \mathcal{R}_3(\mathbb{C})$, consider the most natural candidate as a norm on $\mathcal{R}_n(\mathbb{C})$, $\left\| \left\langle \begin{array}{ccc} & a_1 & \\ a_2 & a_3 & a_4 \\ & & a_5 \end{array} \right\rangle \right\| = \max |a_i|$. Let $R_1 = \left\langle \begin{array}{ccc} & 1 & \\ 1 & 2 & 1 \\ & & 1 \end{array} \right\rangle$ and $R_2 = \left\langle \begin{array}{ccc} & 5 & \\ 1 & 1 & 1 \\ & & 1 \end{array} \right\rangle$. Then $\|R_1 R_2\|_\infty = 11$, whereas $\|R_1\|_\infty \|R_2\|_\infty = 10$. Thus submultiplicativity fails. As already noted in the Corollary 2.4.3, $\mathcal{R}_n(\mathbb{C})$ has only one maximal ideal, that is,

$$\mathcal{I} = \{R \in \mathcal{R}_n(\mathbb{C}) : h(R) = 0\}.$$

Using the one-to-one correspondence between the maximal ideals of a Banach algebra and its multiplicative linear functionals, we conclude the following.

Theorem 3.6. $\mathcal{R}_n(\mathbb{C})$ admits only one multiplicative linear functional, viz. $\varphi : \mathcal{R}_n(\mathbb{C}) \rightarrow \mathbb{C}$, defined by $\varphi(R) = h(R)$, ($R \in \mathcal{R}_n(\mathbb{C})$).

Acknowledgements

The author is grateful to the Dharmsinh Desai University for all research facilities. The author gratefully acknowledges the suggestions made by the referees improving the readability of the paper.

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